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Intuitionistic Fuzzy Hilbert Space And Some Properties

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Abstract: In this paper we introduce Intuitionistic Fuzzy Hilbert space (IFHS) and investigate certain some properties of IFHS. Also, we have given several definitions of Intuitionistic Fuzzy Orthogonal and Intuitionistic Fuzzy Orthonormal discuss in detail.

Key Words: IntuitionisticFuzzy Norm(IFN), Intuitionistic Fuzzy Inner Product Space(IFIP-space), Intuitionistic Fuzzy Hilbert Space(IFH-space), Intuitionistic Fuzzy Orthogonal(IF- orthogonal), Intuitionistic Fuzzy Orthonormal(IF-. Orthonormal).

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I. INTRODUCTION

In 1984 Katsaras [2]first introduced the notation of fuzzy norm,Later Cheng and Mordeson[3], Felbin[4], S.K.Samanta [15] etc.,have defined fuzzy norm and its properties in various ways. R.Biswas[5] and A.M.El-Abyed&Hassan M.EI-Hamouly [6] were the first who tried to give a meaningful definition of fuzzy inner product space (FIP-space) and fuzzy norm function.Later on, a modification of the definition given by R.Biswas was done J.K.Kohli and R.Kumar [7] & [8] in 1995.In 2007 P.Majumdar and S.K.Samanta [10] have given a definition of fuzzy inner product space (FIP-space) whose associated fuzzy norm is of Bag and S.K.Samanta [11],[12] type and some properties. In1986,Atanassov[13] introduced the concept of intuitionistic fuzzy sets, which is again a generalization of fuzzy sets.The concept of intuitionistic fuzzy metric spaces was first studied by J.H.Park. Later R.Saadati [14] has defined intuitionistic fuzzy metric and norm.In 2007, P.Majumdar and S.K.Samanta [15] have given a definition of intuitionistic fuzzy normed linear space. In this paper we consider a intuitionistic fuzzy inner product space (IFIP-space) and introduce the definition of intuitionistic fuzzy Hilbert space (IFH-space). We establish some properties of IFHS. The organization of this paper is as follows:

In section 2, some definitions and preliminary results are given which are used in this paper. In section 3 we introduced the concept of intuitionistic fuzzyHilbert space (IFH-space),intuitionistic fuzzy orthogonal,intuitionistic fuzzy orthonormal and its some properties.

II. PRELIMINARIES

Definition (2.1): [15]
Let U be a linear space over the real or complex field F.Let N and M be two fuzzy subsets of U × R such that $\forall\, x, y\ \in U\ and\ c \in F$ the following holds:
(IN1) N(x,t)+ M(x,t)≤ 1
(IN2)$(\forall\ t > 0, N(x,t) = 1\ ) iff\ x = \underline{0}$
(IN3) $\forall\ t \in R\ and\ t > 0, N(cx,t) = N\left(x, \frac{t}{|c|}\right), if\ c \neq 0$
(IN4) $\forall\ s,t \in R,\ \ x,u \in U, N(x+y, s+t) \geq N(x,s) \wedge N(y,t)$
(IN5) N(x,∗) is non-decreasing function of R and $\lim_{t\to\infty} N(x,t) = 1$
(IN6) $\forall\ t \leq 0, M(x,t) = 1$
(IN7) $(\forall\ t > 0, M(x,t) = 0\ \ ) iff\ x = \underline{0}$
(IN8) $\forall\ t \in R\ and\ t > 0, M(cx,t) =\ M\left(x, \frac{t}{|c|}\right), if\ c \neq 0$
(IN9) $\forall s,t \in R, x,u \in U, M(x+y, s+t) \leq M(x,s) \vee M(y,t)$
(IN10) M(x,∗) is non-increasing function of R.
The pair (N, M) is called an intuitionistic fuzzy norm on U and the triplet (U, N, M) is called an intuitionistic fuzzy normed linear space(IFNLS).
Note 2.1:[15]

(N,M) is an intuitionistic fuzzy norm on U in the sense that associated to $x \in U$ and $t \in R$, $N(x,t)$ indicates the grade of the statement, "the norm of x ≤ t ", and which belongs to [0,1], whereas $M(x,t)$ indicates the grade of the statement, "the norm of $x \nleq t$ and which also belongs to [0,1].

Theorem 2.1: [15]

Let (N,M) be an intuitionistic fuzzy norm on a linear space U. Assume that the following property holds:

(IN11) (M(x,t)< 1 $\forall\, t > 0$) $\Rightarrow$ $x = \underline{0}$

Define $\|x\|_\alpha^N = \wedge\{ t > 0; N(x,t) \geq \alpha\}$ and $\|x\|_\alpha^M = \wedge\{ t > 0; M(x,t) \leq 1-\alpha\}, \alpha \in (0,1).$ Then $\{(\|\ \|_\alpha^N, \|\ \|_\alpha^M): \alpha \in (0,1)\}$ forms a family of ascending pair of norms on liner space U satisfying the inclusion condition: for each $\alpha \in (0,1), \|x\|_\alpha^N \geq \|x\|_\alpha^M$.

Theorem 2.2 : [15]

Let $\{(\|\ \|_\alpha^N, \|\ \|_\alpha^M);\ \alpha \in (0,1)\}$ be a family of a pair of ascending norms on a linear space U satisfying the inclusion condition.

Let N', M': U× $R \to I = [0,1]$ be defined as follows:

$$N'(x,t) = \begin{cases} \vee \ \{ \alpha \in (0,1): \|x\|_\alpha^N \leq t\} & when\ (x,t) \neq (\underline{0},0) \\ 0 & when\ (x,t) \neq (\underline{0},0) \end{cases}$$

$$M'(x,t) = \begin{cases} \wedge \ \{1-\alpha : \|x\|_\alpha^M \leq t\} & when\ (x,t) \neq (\underline{0},0) \\ 1 & when\ (x,t) \neq (\underline{0},0) \end{cases}$$

Then (N',M') is an intuitionistic fuzzy norm on U.

Definition 2.2 : [15]

Let (U, N, M) be an IFNL – space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in U is said to be $\alpha -$ convergent sequence in U if $\exists\, x \in U$, such that $\lim_{n\to\infty} N(x_n - x, t) > \alpha\ \ \forall\, t > 0$ and $\lim_{n\to\infty} M(x_n - x, t) \leq 1-\alpha$, $t > 0$, and x is called the limit of $\{x_n\}$.

Proposition 2.1: [15]

Let (U, N, M) be an IFNL –space satisfying (theorem.2.1).If $\{x_n\}$ be an $\alpha -$ convergent sequence in (U,N,M) then $\|x_n - x\|_\alpha^N \to 0$ as $n \to \infty$ and $\|x_n - x\|_\alpha^M \to 0$ as $n \to \infty$ ($\|\ \|_\alpha^N$ denotes $\alpha -$ norm of N, and $\|\ \|_\alpha^M$ denotes $\alpha -$ norm of M).

Definition2.3: [16] Intuitionistic Fuzzy Inner Product Space (IFIP-space)

Let V be a linear space over the field C of complex numbers.

let μ and $\vartheta: V \times V \times C \to I = [0,1]$ be two mappings such that the following holds $\forall x, y \in V$ and $s,t \in C$,

(IFIP1) μ (x,y,t)+ ϑ(x,y,t) ≤ 1

(IFIP2) $\mu(x+y, z, |t|+|s|) \geq \mu(x,z,|t|) \wedge \mu(y,z,|s|)$

(IFIP3) $\mu(x,y,|st|) \geq \mu(x,x,|s|^2) \wedge \mu(y,y,|t|^2)$

(IFIP4) μ (x,y,t)= μ (y,x,t)

(IFIP5) $\mu(\alpha x, y, t) = \mu\left(x, y, \frac{t}{|\alpha|}\right)$, if $\alpha (\neq 0) \in C$

(IFIP6) (μ (x,x,t)=1 $\forall\, t > 0$) iff $x = \underline{0}$

(IFIP7) μ (x,x,*): $R \to I = [0,1]$ is a monotonic non decreasing function and $\lim_{t\to\infty} \mu(x,x,t) = 1$

(IFIP8) $\vartheta(x+y, |t|+|s|) \leq \vartheta(x,z,|t|) \vee \vartheta(y,z,|s|)$

(IFIP9) $\vartheta(x,y,z,|st|) \leq \vartheta(x,x,|s|^2) \vee \vartheta(y,y,|t|^2)$

(IFIP10) ϑ (x,y,t) = ϑ (y,x,t)

(IFIP11) $\vartheta(\alpha x, y, t) = \vartheta\left(x, y, \frac{t}{|\alpha|}\right)$, if $\alpha (\neq 0) \in C$

(IFIP12) $\forall\ t \in C \backslash R^+$, ϑ (x,x,t)=1

(IFIP13) ϑ(x,x,*) : $R \to I = [0,1]$ is a monotonic non increasing function and $\lim_{t\to\infty} \vartheta(x,x,t) = 0$

Then the pair (μ, ϑ) will be called an intuitionistic fuzzy inner product space(IFIP-space) and we shall call the triple (V, μ, ν) an intuitionistic fuzzy inner product space.

Note 2.2: [16]

(μ, ϑ) is an intuitionistic fuzzy inner product on V in the sence that associated to $x, y \in V$ and $t \in C$, $\mu(x,y,t)$ is grade of statement, "the inner product of x and y is equal to t",and which belongs to [0,1], where as $\vartheta(x,y,t)$ indicates the grade of statement, "the inner product of x and y is not equal to t" and which also belongs to [0,1].

Note 2.3: [16]

We may observe that the property:

$(\mu(x,x,t) = 0, \forall\, t \leq 0$ and $x \in V$) holds from (IFIP1)and(IFIP12). similarly from (IFIP1)and (IFIP6) implies ((ϑ (x,x,t)=0 $\forall\, t > 0$) $\Leftarrow x = \underline{0}$).

Theorem 2.3:[16]

Let (μ, ϑ) be an IFIP on V. Then if we define two functions $N, M: V \times R \to I = [0,1]$ as follows:

$$N(x,t)=\begin{cases}\mu(x,x,t^2)\forall t\in R\ and\ t>0\\ 0\qquad\qquad \forall t\in R\ and\ t\le 0\end{cases}$$
$$M(x,t)=\begin{cases}\vartheta(x,x,t^2)\forall t\in R\ and\ t>0\\ 1\qquad\qquad \forall t\in R\ and\ t\le 0\end{cases}$$

Then (N,M) is an intuitionistic fuzzy norm on V of the type given in [6].

Propositon 2.2: [16]

If

(μ_1,ϑ_1) *and* (μ_2,ϑ_2) are twointuitionistic fuzzy inner product on V , then $(\mu,\vartheta)=(\mu_1\wedge\ \mu_2,\vartheta_1\vee\ \vartheta_2)$ *is* also, an IFIP on V.

Proposition 2.3: [16]

Let (V,μ,ϑ) be an IFIP space then

(i) For $x,y,z\in V\ and\ t,s\in C,\mu\,(x,y+z,|t|+|s|)$
$$\geq\ \mu(x,y,|t|)\bigwedge\ \mu(x,z,|s|)\ and\ \vartheta(x,y+z,|t|+|s|)$$
$$\leq\ \vartheta(x,y,|t|)\bigvee\ \vartheta(x,z,|s|)$$

(ii) For $\lambda\in C\ and\ \lambda\neq 0,\mu(x,\lambda y,t)=\mu\,(\lambda x,y,t)\ and\ \vartheta(x,\lambda y,t)=\ \vartheta\,(\lambda x,y,t)$

Remark 2.1: [16]

1. Assume that ϑ satisfies the following condition:**(IFIP14)** (ϑ (x,x,t^2)$<1\ \forall\ t>0) \Rightarrow x=\underline{0}$ then from(IFIP1) and(IFIP14), it follows that**(IFIP14a)** (μ (x,x,t^2)$>0\ \ \forall\ t>0\ )\Rightarrow x=\underline{0}$.

2. By the theorem 2.3 and decomposition theorem 1.1 and 1.2 we have the following:
Let (V,μ,ϑ) *be an IFIP space satisfying* $(IFIP14)$*. Define* $\forall\ \alpha\in(0,1)$, $\|x\|_\alpha^N\ =\ \wedge\{\ t>0;\mu(x,x,t^2)\geq \alpha$ *and* $\|x\|_\alpha^M=\wedge\ \{t>0;\vartheta(x,x,t^2)\leq 1-\alpha\}$ *then* $\{(\|\ \|_\alpha^N,\|\ \|_\alpha^M);\alpha\in(0,1)\}$ is an ascending family of pair of crisp norms on V generated from (μ,ϑ) *and which* satisfies the inclusion condition(IC): Because, $t>0,\mu(x,x,t^2)\geq\alpha\Rightarrow \vartheta(x,x,t^2)\leq 1-\alpha$

$$\Rightarrow\{\ t>0:\ \mu(x,x,t^2)\geq\alpha\}\ \subset\{\ t>0;\vartheta(x,x,t^2)\leq 1-\alpha\}$$
$$\Rightarrow\|x\|_\alpha^N\ =\wedge\{\ t>0;\mu(x,x,t^2)\geq\alpha\}\ \geq\ \wedge\{\ t>0;\vartheta(x,x,t^2)\leq 1-\alpha\}=\|x\|_\alpha^M$$
$$\Rightarrow\|x\|_\alpha^N\ \geq\|x\|_\alpha^M\ \ \forall\alpha\in(0,1)$$

In the sequel we shall consider the following conditions:

$(\boldsymbol{IFIP15})\ \forall\ x,y\in V\ and\ p,q\in R\ ,\mu(x+y,x+y,2q^2)\wedge\mu(x-y,x-y,2p^2)\ \geq\mu(x,x,p^2)\wedge\mu(y,y,q^2)$ *and* $\big((\boldsymbol{IFIP15a})\ \vartheta(x+y,x+y,2q^2)\vee\vartheta(x-y,x-y,2p^2)\ \leq\ \vartheta(x,x,p^2)\vee\vartheta(y,y,q^2)$.

Proposition 2.4: [16] (Parallelogram law)

Let **(μ,ϑ)** be an IFIP space on a linear space V over the field of complex numbers satisfying (IFIP14),(IFIP15) and (IFIP15 a).
Let $\alpha\in(0,1)$ *and* $(\|\ \ \|_\alpha^N,\|\ \ \|_\alpha^M)$ be the α norms generated from IFIP(μ,ϑ) *on V. Then*
$\|x-y\|_\alpha^{N\,2}+\|x+y\|_\alpha^{N\,2}=2\Big(\|x\|_\alpha^{N\,2}+\|y\|_\alpha^{N\,2}\Big)$ and $\|x-y\|_\alpha^{M\,2}+\|x+y\|_\alpha^{M\,2}=2\Big(\|x\|_\alpha^{M\,2}+\|y\|_\alpha^{M\,2}\Big)$

3. Main Results

Definition (3.1): ($\boldsymbol{l-Complete}$)

Let (V,μ,ϑ) be an IFIP space satisfyingRemark 2.1(IFIPS 14, 14a, 15,15a). V is said to be level complete ($l-Complete$) if for any $\alpha\in(0,1)$, every Cauchy sequence converges in V w.r.t. the α – norms $\|\ \ \|_\alpha^N$ and $\|\ \ \|_\alpha^M$ generated by IFN (N,M) which is induced by IFIP (μ,ϑ).

Definition (3.2): (IFHS)

Let (V,μ,ϑ) be an IFIP space. V is said to be IFHS if it is $l-Complete$.

Definition (3.3): ($\boldsymbol{\alpha}$ – IF Orthogonal)

Let $\alpha\in(0,1)$ and (V,μ,ϑ) be an IFIP space satisfyingRemark 2.1(IFIPS 14,14a,15,15a). Now if $x,y\in V$ be such that $\langle x,y\rangle_\alpha^N=0$ and $\langle x,y\rangle_\alpha^M=0$, then we say that x,y are α – IF orthogonal to each other and it is denoted by $x\perp_\alpha^N y$ and $x\perp_\alpha^M y$.

Let A be a subset of V and $x\in V$. Now if $\langle x,y\rangle_\alpha^N=0\ \forall\ y\in A$ and $\langle x,y\rangle_\alpha^M=0\ \forall\ y\in A$, then we say that x is α – IF orthogonal to A. It is denoted by $x\perp_\alpha^N A$ and $x\perp_\alpha^M A$.

Definition (3.4): (IF Orthogonal)

Let (V,μ,ϑ) be an IFIP space satisfyingRemark 2.1(IFIPS 14,14a,15,15a). Now if $x,y\in V$ be such that $\langle x,y\rangle_\alpha^N=0$ and $\langle x,y\rangle_\alpha^M=0$, $\forall\ \alpha\in(0,1)$, then we say that x,y are intuitionistic fuzzy orthogonal to each other and it is denoted by $x\perp y$.Thus $x\perp y$ iff $x\perp_\alpha^N y\ \&\ x\perp_\alpha^M y\ \forall\ \alpha\in(0,1)$.

Definition (3.5): ($\boldsymbol{\alpha}$ – IF Orthonormal)

Let (V,μ,ϑ) be an IFIP space satisfying Remark 2.1(IFIPS 14,14a,15,15a) and $\alpha\in(0,1)$. An α – IF orthogonal set A in V is said to be α – IF orthonormal if the elements have α- norm 1 that is $\forall\ x,y\in A$,

$$\langle x,y\rangle_\alpha^N=\begin{cases}1 & if\ x=y\\ 0 & if\ x\neq y\end{cases}\&\langle x,y\rangle_\alpha^M=\begin{cases}1 & if\ x=y\\ 0 & if\ x\neq y\end{cases}$$

where $\langle , \rangle_\alpha^N$ and $\langle , \rangle_\alpha^M$ are the induced inner product by (μ, ϑ).

Definition (3.6): (IF Orthonormal)

Let (V, μ, ϑ) be an IFIP space satisfyingRemark 2.1(IFIPS 14,14a,15,15a). A IF orthogonal set A in V is said to be IF orthonormal, if the elements have α- norm 1 $\forall\, \alpha \in (0,1)$ that is $\forall\, x, y \in A$,

$$\langle x, y\rangle_\alpha^N = \begin{cases} 1 & if\ x = y \\ 0 & if\ x \neq y \end{cases} \& \langle x, y\rangle_\alpha^M = \begin{cases} 1 & if\ x = y \\ 0 & if\ x \neq y \end{cases}$$

where $\langle , \rangle_\alpha^N$ and $\langle , \rangle_\alpha^M$ are the induced inner product by (μ, ϑ).

Note 3.1:

An α – IF orthonormal set and IF orthonormal set in an IFIP space are linearly independent.

Proposition (3.1):

Let $\{e_k\}$ be anIF orthonormal sequence in an IFHS (V, μ, ϑ) satisfying Remark 2.1(IFIPS 14,14a,15,15a). Then the series $\sum_k \alpha_k e_k$ converges (w.r.t.$\|\ \ \|_\alpha^N$ and $\|\ \ \|_\alpha^M; \alpha \in (0,1)$, where $\|\ \ \|_\alpha^N$ and $\|\ \ \|_\alpha^M$are the α- norms of (*N*, *M*) which are induced by (μ, ϑ)) iff $\sum_k |\alpha_k|^2$ converges w.r.t. $\|\ \ \|_\alpha^N$ and $\|\ \ \|_\alpha^M$.

Proof:

Let $S_n = \sum_{k=1}^n \alpha_k e_k$ and $\sigma_n = \sum_{k=1}^n |\alpha_k|^2$. Then

$$\|S_n - S_m\|_\alpha^{N^2} = \langle \sum_{k=1}^n \alpha_k e_k - \sum_{k=1}^m \alpha_k e_k, \sum_{k=1}^n \alpha_k e_k - \sum_{k=1}^m \alpha_k e_k \rangle_\alpha^N$$

$$\& \|S_n - S_m\|_\alpha^{M^2} = \langle \sum_{k=1}^n \alpha_k e_k - \sum_{k=1}^m \alpha_k e_k, \sum_{k=1}^n \alpha_k e_k - \sum_{k=1}^m \alpha_k e_k \rangle_\alpha^M$$

$i.e.$ $\quad \|S_n - S_m\|_\alpha^{N^2} = |\alpha_{m+1}|^2 + |\alpha_{m+2}|^2 + \cdots + |\alpha_n|^2$

$\& \|S_n - S_m\|_\alpha^{M^2} = |\alpha_{m+1}|^2 + |\alpha_{m+2}|^2 + \cdots + |\alpha_n|^2$

$i.e.$ $\|S_n - S_m\|_\alpha^{N^2} = \sigma_n - \sigma_m \& \|S_n - S_m\|_\alpha^{M^2} = \sigma_n - \sigma_m \forall\, \alpha \in (0,1)$

Hence S_n is a Cauchy sequence w,r,t. $\|\ \ \|_\alpha^N$ and $\|\ \ \|_\alpha^M$ $\forall\, \alpha \in (0,1)$ iff σ_n is Cauchy in R(set of all real numbers). Hence S_n is Cauchy iff σ_n is Cauchy in R.

Note 3.2:

In case of α – IF orthonormal sequence the proposition holds.

Proposition (3.2):

Let (V, μ, ϑ) be an IFHS satisfyingRemark 2.1(IFIPS 14,14a,15,15a) and $\alpha \in (0,1)$ and let $\{e_k\}$ be an α - IF orthonormal sequence in V. If the series $\sum_{k=1}^\infty \beta_k e_kand\sum_{k=1}^\infty \beta_k^* e_k$is α-convergent w.r.t. (*N*, *M*) induced by (μ, ϑ), then the coefficients $\beta_k = \langle x, e_k\rangle_\alpha^N$ and $\beta_k^* = \langle x, e_k\rangle_\alpha^M$ denotes the sums $\sum_{k=1}^\infty \beta_k e_k \& \sum_{k=1}^\infty \beta_k^* e_k$ and hence $x = \sum_{k=1}^\infty \langle x, e_k\rangle_\alpha^N e_k \& x = \sum_{k=1}^\infty \langle x, e_k\rangle_\alpha^M e_k$.

Proof:

Since $\sum_{k=1}^\infty \beta_k e_k$ and $\sum_{k=1}^\infty \beta_k^* e_k$ are α-convergent w.r.t. $\|\ \ \|_\alpha^N$ and $\|\ \ \|_\alpha^M$(def 2.2 & Prop2.1)

$$i.e. \left\|\sum_{k=1}^n \beta_k e_k - x\right\|_\alpha^N \to 0\ as\ n \to \infty \quad \& \left\|\sum_{k=1}^n \beta_k^* e_k - x\right\|_\alpha^M \to 0\ as\ n \to \infty$$

Let $S_n = \sum_{k=1}^\infty \beta_k e_k$ and $S_n^* = \sum_{k=1}^\infty \beta_k^* e_k$.

Taking IP withS_n, S_n^* and e_j and using the definition of α – IF orthogonality, we have

$$\langle S_n, e_j\rangle_\alpha^N = \beta_j, \langle S_n^*, e_j\rangle_\alpha^M = \beta_j^* \qquad for\ j = 1,2, \ldots, k.$$

$\therefore S_n \to x$ w.r.to.$\|\ \ \|_\alpha^N$ and $S_n^* \to x$ w.r.to.$\|\ \ \|_\alpha^M$

$\Rightarrow \langle S_n, e_j\rangle_\alpha^N \to \langle x, e_j\rangle_\alpha^N = \beta_j$ and $\langle S_n^*, e_j\rangle_\alpha^M \to \langle x, e_j\rangle_\alpha^M = \beta_j^*$

$\therefore x = \sum_{k=1}^\infty \beta_k e_k = \sum_{k=1}^\infty \langle x, e_k\rangle_\alpha^N e_k$ & $x = \sum_{k=1}^\infty \beta_k^* e_k = \sum_{k=1}^\infty \langle x, e_k\rangle_\alpha^M e_k$

Hence the proof.

Theorem (3.1):

Let (V, μ, ϑ) be an IFHS satisfyingRemark 2.1(IFIPS 14,14a,15,15a) and let $\{e_k\}$ be anIF orthonormal sequence in V. If the series $\sum_{k=1}^\infty \gamma_k e_k$ and $\sum_{k=1}^\infty \gamma_k^* e_k$ converge w.r.t. (*N*, *M*) induced by (μ, ϑ), then

$\gamma_k = \langle x, e_k\rangle_\alpha^N = \langle x, e_k\rangle_\beta^N \ \& \gamma_k^* = \langle x, e_k\rangle_\alpha^M = \langle x, e_k\rangle_\beta^M,\ \forall\, \alpha, \beta \in (0,1)$

Where $\langle , \rangle_\alpha^N$ and $\langle , \rangle_\alpha^M$ denote α- IP induced by (μ, ϑ), x denotes the sum of $\sum_{k=1}^\infty \gamma_k e_k \& \sum_{k=1}^\infty \gamma_k^* e_k$.

Hence $x = \sum_{k=1}^\infty \langle x, e_k\rangle_\alpha^N e_k = \sum_{k=1}^\infty \langle x, e_k\rangle_\beta^N e_k$ & $x = \sum_{k=1}^\infty \langle x, e_k\rangle_\alpha^M e_k = \sum_{k=1}^\infty \langle x, e_k\rangle_\beta^M e_k \quad \forall\, \alpha, \beta \in (0,1)$

Proof:

Since $\{e_k\}$ is IF orthonormal. it is orthonormal w.r.to. each $\langle , \rangle_\alpha^N$ and $\langle , \rangle_\alpha^M$, $\alpha \in (0,1)$.

Now $\sum_{k=1}^\infty \gamma_k e_k$ and$\sum_{k=1}^\infty \gamma_k^* e_k$is convergent w.r.to. (*N*, *M*) implies they are convergent w.r.t.$\|\ \ \|_\alpha^N$ and $\|\ \ \|_\alpha^M$, $\alpha \in (0,1)$ [by Proposition(3.1)]. We have

$$\langle\sum_{k=1}^{n}\gamma_k e_k\,,e_j\rangle_\alpha^N=\gamma_j=\langle\sum_{k=1}^{n}\gamma_k e_k\,,e_j\rangle_\beta^N\,\&\,\langle\sum_{k=1}^{n}\gamma_k^* e_k\,,e_j\rangle_\alpha^M=\gamma_j^*=\langle\sum_{k=1}^{n}\gamma_k^* e_k\,,e_j\rangle_\beta^M\,,j=1,2,\dots,k$$

Taking limit as $n\to\infty$, we have

$$\lim_{n\to\infty}\langle\sum_{k=1}^{n}\gamma_k e_k\,,e_j\rangle_\alpha^N=\lim_{n\to\infty}\gamma_j=\lim_{n\to\infty}\langle\sum_{k=1}^{n}\gamma_k e_k\,,e_j\rangle_\beta^N\,\&\lim_{n\to\infty}\langle\sum_{k=1}^{n}\gamma_k^* e_k\,,e_j\rangle_\alpha^M=\lim_{n\to\infty}\gamma_j^*=\lim_{n\to\infty}\langle\sum_{k=1}^{n}\gamma_k^* e_k\,,e_j\rangle_\beta^M$$

$$\langle\sum_{k=1}^{\infty}\gamma_k e_k\,,e_j\rangle_\alpha^N=\gamma_j=\langle\sum_{k=1}^{\infty}\gamma_k e_k\,,e_j\rangle_\beta^N\,\&\,\langle\sum_{k=1}^{\infty}\gamma_k^* e_k\,,e_j\rangle_\alpha^M=\gamma_j^*=\langle\sum_{k=1}^{\infty}\gamma_k^* e_k\,,e_j\rangle_\beta^M$$

[Since $\langle\,,\,\rangle_\alpha^N$ and $\langle\,,\,\rangle_\alpha^M$, are continuous $\forall\,\alpha\in(0,1)$]

$$\langle x,e_j\rangle_\alpha^N=\gamma_j=\langle x,e_j\rangle_\beta^N\,\&\,\langle x,e_j\rangle_\alpha^M=\gamma_j^*=\langle x,e_j\rangle_\beta^M\quad\forall\,\alpha,\beta\in(0,1),\qquad j=1,2,3,\dots$$

Hence the proof

Definition (3.7): (Complete IF Orthonormal Set)

Let (V,μ,ϑ) be an IFIP space satisfies Remark 2.1(IFIPS 14,14a,15,15a).An IF orthonormal set $A\subset V$ is called complete IF orthonormal set if there is no α – IF orthonormal set ($\alpha\in(0,1)$) of which A is a proper subset. If A is countable then it is called complete IF orthonormal sequence.

Theorem (3.2): (Bessel's Inequality)

Let (V,μ,ϑ) be an IFIP space satisfiesRemark 2.1(IFIPS 14,14a,15,15a), $\alpha\in(0,1)$ and $\{e_k\}$ *be an* α – IF orthonormal sequence in V. Then for every $x\in V$,

$$\sum_{k=1}^{\infty}|\langle x,e_k\rangle_\alpha^N|^2\le{\|x\|_\alpha^N}^2\,\&\sum_{k=1}^{\infty}|\langle x,e_k\rangle_\alpha^M|^2\le{\|x\|_\alpha^M}^2$$

Proof:

Since α – IF orthonormal sequence is orthonormal sequence in (V, $\langle\,,\,\rangle_\alpha^N$, $\langle\,,\,\rangle_\alpha^M$) so by Bessel's inequality in crisp inner product we have

$$\sum_{k=1}^{\infty}|\langle x,e_k\rangle_\alpha^N|^2\le{\|x\|_\alpha^N}^2\,\&\sum_{k=1}^{\infty}|\langle x,e_k\rangle_\alpha^M|^2\le{\|x\|_\alpha^M}^2\ \ \forall\,\alpha\in(0,1)$$

Theorem (3.3):

Let (V,μ,ϑ) be a Hilbert space satisfying Remark 2.1(IFIPS15,15a) and $\{e_i\}$ is an IF orthonormal sequence in V then the following statements are equivalent.

(i) $\{e_i\}$ is complete IF orthonormal.

(ii) $x\perp e_i$ for $i=1,2,\dots\Rightarrow x=\underline{0}$.

(iii) For every $x\in V$, $x=\sum_{i=1}^{\infty}\langle x,e_i\rangle_\alpha^N e_i\,\&\quad x=\sum_{i=1}^{\infty}\langle x,e_i\rangle_\alpha^M e_i\ \forall\,\alpha\in(0,1)$ and hence $\langle x,e_k\rangle_\alpha^N=\langle x,e_k\rangle_\beta^N\&\langle x,e_k\rangle_\alpha^M=\langle x,e_k\rangle_\beta^M\quad\forall\,\alpha,\beta\in(0,1)$. i.e. x is independent on α.

(iv) For every $x\in V$, ${\|x\|_\alpha^N}^2=\sum_{i=1}^{\infty}|\langle x,e_i\rangle_\alpha^N|^2\&{\|x\|_\alpha^M}^2=\sum_{i=1}^{\infty}|\langle x,e_i\rangle_\alpha^M|^2\ \forall\,\alpha\in(0,1)$ and hence ${\|x\|_\alpha^N}^2={\|x\|_\beta^N}^2\&{\|x\|_\alpha^M}^2={\|x\|_\beta^M}^2$.

Proof:

(a) Suppose (i) holds.

Let $\{e_i\}$ be a complete IF orthonormal sequence and $x\perp e_i$ for $i=1,2,\dots$

$\Rightarrow x\perp_\alpha^N e_i\,\&\quad x\perp_\alpha^M e_i\ \ \forall\,\alpha\in(0,1),\,i=1,2,\dots$

$\Rightarrow\langle x,e_i\rangle_\alpha^N=0\quad\&\langle x,e_i\rangle_\alpha^M=0\ \ \forall\,\alpha\in(0,1)$

Set for fixed α_0, $e^{\alpha_0}=\frac{x}{\|x\|_{\alpha_0}^N}$.

Then ${\|e^{\alpha_0}\|_{\alpha_0}^N}^2=\langle e^{\alpha_0},e^{\alpha_0}\rangle_{\alpha_0}^N=1\ \ \&\langle e^{\alpha_0},e_i\rangle_{\alpha_0}^N=0\ for\ i=1,2,\dots\dots$

$$\&\quad e^{\alpha_0}=\frac{x}{\|x\|_{\alpha_0}^M}\ Then\ {\|e^{\alpha_0}\|_{\alpha_0}^M}^2=\langle e^{\alpha_0},e^{\alpha_0}\rangle_{\alpha_0}^M=1\ \ \&\langle e^{\alpha_0},e_i\rangle_{\alpha_0}^M=0\ for\ i=1,2,\dots\dots$$

$\therefore$ we get an α_0- IF orthonormal sequence $\{e^{\alpha_0},e_1,e_2,\dots\}$ of which $\{e_1,e_2,\dots\}$ is proper subset – a contraction to completeness.

$\therefore e^{\alpha_0}=0\Rightarrow x=\underline{0}$

So (i) ⇒ (ii).

(b) Suppose (ii) holds.

Let $x\perp e_i$ for $i=1,2,\dots\Rightarrow x=0$.

$$\Rightarrow x-\sum_{i=1}^{\infty}\langle x,e_i\rangle_\alpha^N e_i\perp_\alpha^N e_j\,\&\quad x-\sum_{i=1}^{\infty}\langle x,e_i\rangle_\alpha^M e_i\perp_\alpha^M e_j\,,\qquad\forall\,\alpha\in(0,1)$$

$$\Rightarrow x - \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^N e_i = 0 \quad \& \quad x - \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^M e_i = 0$$

$$\Rightarrow x = \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^N e_i = \sum_{i=1}^{\infty} \langle x, e_i \rangle_\beta^N e_i \;\; \& \quad x = \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^M e_i = \sum_{i=1}^{\infty} \langle x, e_i \rangle_\beta^M e_i \qquad [\,since\ x = 0]$$

$$\Rightarrow \sum_{i=1}^{\infty} \left(\langle x, e_i \rangle_\alpha^N - \langle x, e_i \rangle_\beta^N \right) e_i = 0 \;\; \& \sum_{i=1}^{\infty} \left(\langle x, e_i \rangle_\alpha^M - \langle x, e_i \rangle_\beta^M \right) e_i = 0$$

Since $\{e_i\}$ is linearly independent, therefore

$\langle x, e_i \rangle_\alpha^N - \langle x, e_i \rangle_\beta^N = 0 \quad \& \langle x, e_i \rangle_\alpha^M - \langle x, e_i \rangle_\beta^M = 0 \;\; \forall\, \alpha, \beta \in (0,1)$

$\Rightarrow \langle x, e_i \rangle_\alpha^N = \langle x, e_i \rangle_\beta^N \& \langle x, e_i \rangle_\alpha^M = \langle x, e_i \rangle_\beta^M$

Thus (ii) ⇒ (iii).

(c) Suppose (iii) holds.

Let $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^N e_i \;\&\quad x = \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^M e_i \quad \forall\, \alpha \in (0,1)$

Now $\|x\|_\alpha^{N^2} = \langle x, x \rangle_\alpha^N \& \|x\|_\alpha^{M^2} = \langle x, x \rangle_\alpha^M$

$$\Rightarrow \|x\|_\alpha^{N^2} = \langle \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^N e_i \,, \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^N e_i \rangle_\alpha^N \;\& \|x\|_\alpha^{M^2} = \langle \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^M e_i \,, \sum_{i=1}^{\infty} \langle x, e_i \rangle_\alpha^M e_i \rangle_\alpha^M$$

$$\Rightarrow \lim_{n\to\infty} \|x\|_\alpha^{N^2} = \lim_{n\to\infty} \langle \sum_{i=1}^{n} \langle x, e_i \rangle_\alpha^N e_i \,, \sum_{i=1}^{n} \langle x, e_i \rangle_\alpha^N e_i \rangle_\alpha^N \;\& \lim_{n\to\infty} \|x\|_\alpha^{M^2} = \lim_{n\to\infty} \langle \sum_{i=1}^{n} \langle x, e_i \rangle_\alpha^M e_i \,, \sum_{i=1}^{n} \langle x, e_i \rangle_\alpha^M e_i \rangle_\alpha^M$$

$$\Rightarrow \|x\|_\alpha^{N^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \langle x, e_i \rangle_\alpha^N \, \overline{\langle x, e_i \rangle_\alpha^N} \& \|x\|_\alpha^{M^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \langle x, e_i \rangle_\alpha^M \, \overline{\langle x, e_i \rangle_\alpha^M}$$

$$\Rightarrow \|x\|_\alpha^{N^2} = \sum_{i=1}^{\infty} |\langle x, e_i \rangle_\alpha^N|^2 \& \|x\|_\alpha^{M^2} = \sum_{i=1}^{\infty} |\langle x, e_i \rangle_\alpha^M|^2$$

Now from (iii) we have $\langle x, e_i \rangle_\alpha^N = \langle x, e_i \rangle_\beta^N \& \langle x, e_i \rangle_\alpha^M = \langle x, e_i \rangle_\beta^M \quad \forall\, i = 1,2,3, \dots \quad \forall\, \alpha, \beta \in (0,1)$

Using this we get

$$\Rightarrow \|x\|_\alpha^{N^2} = \sum_{i=1}^{\infty} |\langle x, e_i \rangle_\alpha^N|^2 = \sum_{i=1}^{\infty} \left|\langle x, e_i \rangle_\beta^N\right|^2 = \|x\|_\beta^{N^2} \& \|x\|_\alpha^{M^2} = \sum_{i=1}^{\infty} |\langle x, e_i \rangle_\alpha^M|^2 = \sum_{i=1}^{\infty} \left|\langle x, e_i \rangle_\beta^M\right|^2 = \|x\|_\beta^{M^2}$$

$i.e.\ \|x\|_\alpha^{N^2} = \|x\|_\beta^{N^2} \quad and \quad \|x\|_\alpha^{M^2} = \|x\|_\beta^{M^2} \quad \forall\, \alpha, \beta \in (0,1)$

So (iii) ⇒(iv).

(d) Suppose (iv) holds $\{e_i\}$ is not complete. Then we get for an $\alpha \in (0,1) \{e^\alpha \;, e_1, e_2, \dots\}$ of which $\{e_1, e_2, \dots\}$ is proper subset and $\|e^\alpha\|_\alpha^N = 1, \|e^\alpha\|_\alpha^M = 1$.

$\langle e^\alpha, e_i \rangle_\alpha^N = 0, \langle e^\alpha, e_i \rangle_\alpha^M = 0, \forall\, i = 1,2,3, \dots$

Now $\|e^\alpha\|_\alpha^{N^2} = \sum_{i=1}^{\infty} |\langle e^\alpha, e_i \rangle_\alpha^N|^2 = 0 \quad \& \|e^\alpha\|_\alpha^{M^2} = \sum_{i=1}^{\infty} |\langle e^\alpha, e_i \rangle_\alpha^M|^2 = 0$

$\Rightarrow\ e^\alpha = \bar{0}$

Thus (iv) ⇒(i).

III. CONCLUSION

From this paper, the idea of Intuitionistic Fuzzy Hilbert space (IFHS) is relatively new. We attempted to prove some properties of Intuitionistic Fuzzy Hilbert space and some important concepts viz, definitions of intuitionistic fuzzy orthogonal,intuitionistic fuzzy orthonormal etc., The results of this paper will be helpful for researchers to develop Intuitionistic fuzzy functional analysis.

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