An Interpolation Process on Weighted (0,1,2;0)- Interpolation on Laguerre Polynomial

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Abstract: The aim of this paper is to study a special kind of mixed type of interpolation on the zeros of \( L_n^{(k)}(x) \) and \( L_n^{(0-1)}(x) \) and on the interval \([0,\infty)\) with boundary (Hermite) conditions gives a simultaneous approximation to a differentiable function and its derivative under what conditions. We have investigated the existence, uniqueness, explicit representation and estimation of interpolatory polynomial.

Keywords: lacunary interpolation, Pál - Type interpolation, Laguerre Polynomial

MSC 2000: 41 A 05 65 D 32

I. INTRODUCTION

Pál [6] proved that when the function values are prescribed on one set of \( n \) points and derivative values on other set of \( n-1 \) points, then there exist no unique polynomial of degree \( \leq 2n-2 \), but prescribing function value at one more point not belonging to former set of \( n \) points there exists a unique polynomial of degree \( \leq 2n-1 \). Pál [6] considered an interscaled set of nodes which were the zeros of some polynomial \( P(x) \) and its derivative \( P'(x) \). The weighted lacunary interpolation was studied and modified by mathematicians such as Szili L., Mathur P. and Datta S., Mathur K.K. and Srivastava R., Srivatava R and Mathur K.K. etc [9][4][5][10]. Lénárd M. [3] investigated the Pál – type interpolation problem on the nodes of Laguerre abscissas. The aim of this paper is to study Pál – type interpolational polynomial with \( \omega_{n+k}(x) = x^k L_n^{(k)}(x) \). We have determined the existence, uniqueness and explicit representation of fundamental polynomials of such special kind of mixed type of interpolation on interval \([0,\infty)\). For this we have considered the problem in which \( \{\xi_i\}_{i=1}^n \) and \( \{\xi_i^*\}_{i=1}^n \) the two sets of interscaled nodal points.

\[
0 \leq \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n < \xi_{n+1} < \xi_n < \infty
\]

on the interval \([0,\infty)\). We seek to determine a polynomial \( R_n(x) \) of minimal possible degree \( 4n+k \) satisfying the interpolatory conditions:

\[
R_n(\xi_i) = g_i, \quad R'_n(\xi_i) = g'_i, \quad (\rho R_n)'(\xi_i) = g''_i
\]

\[(1.2) \quad R_n(\xi_i) = d_i^***, \quad \text{for } i = 1(1)n
\]

where \( g_i, g'_i, g''_i \) are arbitrary real numbers. Here Laguerre polynomials \( L_n^{(k)}(x) \) and \( L_n^{(k-1)}(x) \) have zeroes \( \{\xi_i\}_{i=1}^n \) and \( \{\xi_i^*\}_{i=1}^n \) respectively and \( x_0 = 0 \). We prove existence, uniqueness, explicit representation and estimation of fundamental polynomials with respect to the weight function \( \rho(x) = e^{-x/2}x^{(k+1)/2} \).

II. PRELIMINARIES

In this section we shall give some well-known results which are as follows:

As we know that the Laguerre polynomial is a constant multiple of a confluent hypergeometric function so the differential equation is given by

\[
(2.1) \quad xD^2 L_n^{(k)}(x) + (1 + k - x)DL_n^{(k)}(x) + nL_n^{(k)}(x) = 0
\]

\[
(2.2) \quad L_n^{(k-1)}(x) = -L_{n-1}^{(k)}(x)
\]

Also using the identities
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\[
(2.3) \quad l_n^{(k)}(x) = L_n^{(k+1)}(x) - L_{n-1}^{(k+1)}(x)
\]

\[
(2.4) \quad xL_n^{(k)}(x) = nL_n^{(k)}(x) - (n + k)L_{n-1}^{(k)}(x)
\]

We can easily find a relation

\[
(2.5) \quad \frac{d}{dx}[x^kL_n^{(k)}(x)] = (n + k)x^{k-1}L_{n-1}^{(k)}(x)
\]

By the following conditions of orthogonality and normalization, we define Laguerre polynomial \(L_n^{(k)}(x)\), for \(k > -1\).

\[
(2.6) \quad \int_0^\infty e^{-x}x^kL_n^{(k)}(x)L_m^{(k)}(x)dx = \Gamma(k + 1) \left(\frac{n+k}{n}\right) \delta_{nm} \quad n, m = 0, 1, 2, \ldots .
\]

\[
(2.7) \quad L_n^{(k)}(x) = \sum_{\mu=0}^{n+k} \binom{n+k}{\mu} \left(-\frac{x}{n}\right)^{\mu} \frac{\mu!}{n^\mu}
\]

The fundamental polynomials of Lagrange interpolation are given by

\[
(2.8) \quad l_j(x) = \frac{l_n^{(k)}(x)}{l_n^{(k)}(x_j)} = \delta_{ij}
\]

\[
(2.9) \quad l_j'(x) = \frac{l_n^{(k-1)}(x)}{l_n^{(k-1)}(x_j)} = \delta_{ij}
\]

\[
(2.10) \quad l_j''(y_j) = \begin{cases} 
\frac{l_n^{(k-1)}(y_j)}{l_n^{(k-1)}(y_j)} & i \neq j, \\
(\frac{1}{k-y_j}) & i = j
\end{cases}, \quad j = 1(1)n
\]

\[
(2.12) \quad l_j'(y_j) = \frac{1}{(y_j-x_j)} \left(l_n^{(k)}(x_j) - l_n^{(k)}(y_j)(y_j-x_j)\right), \quad j = 1(1)n
\]

For the roots of \(l_n^{(k)}(x)\) we have

\[
(2.13) \quad 2\sqrt{x_j} = \frac{1}{\pi^n}[j\pi + O(1)]
\]

\[
(2.14) \quad \eta(x)|S_n^{(i)}(x)| = O(1), \quad \text{where} \ \eta(x) \ \text{is the weight function}
\]

\[
(2.15) \quad |L_n^{(k)}(x_j)| \sim j^{-\frac{1}{2}n^k + 1}, \quad (0 < x_j \leq \Omega, n = 1, 2, 3, \ldots .)
\]

\[
(2.16) \quad |L_n^k(x)| = \begin{cases} 
\frac{x^{\frac{k-1}{2}}}{\frac{1}{2n^2}} & c^2n^{-1} \leq x \leq \Omega \\
0 & 0 \leq x \leq c^{-1}n
\end{cases},
\]

\[
(2.17) \quad O(l_j(x)) = 0 \left(L_j'(x)\right) = 1
\]

### III. NEW RESULT

**Theorem 3.1:** For \(n\) and \(k\) fixed positive integer let \(\{g_i\}_{i=1}^n\), \(\{g_i^1\}_{i=1}^n\), \(\{g_i^2\}_{i=1}^n\), \(\{g_i^3\}_{i=1}^n\), and \(\{g_i^4\}_{i=1}^n\) are arbitrary real numbers then there exists a unique polynomial \(R_n(x)\) of minimal possible degree \(\leq 4n+k\) on the nodal points (1.1) satisfying the condition (1.2) and (1.3). The polynomial \(R_n(x)\) can be written in the form

\[
(3.1) \quad R_n(x) = \sum_{j=1}^{n} U_j(x)g_j^1 + \sum_{j=1}^{n} V_j(x)g_j^2 + \sum_{j=1}^{n} W_j(x)g_j^3 + \sum_{j=1}^{n} X_j(x)g_j^4 + \sum_{j=0}^{k} C_j(x)g_j^5
\]

where \(U_j(x)\), \(V_j(x)\), \(W_j(x)\), \(X_j(x)\) and \(C_j(x)\) are fundamental polynomials of degree \(\leq 4n+k\) given by
(3.2) \( U_j(x) = \frac{x^{k+1}l_j(x)}{\sum_{j=0}^{n+k} l_n^j(x)} \left[ 1 - (x - x_j) \right] \left[ \frac{1 + k - 5x_j}{2x_j} \right] + c_j (x - x_j) \]

(3.3) \( V_j(x) = \frac{x^{k+1}l_n^j(x)}{\sum_{j=0}^{n+k} l_n^j(x)} \frac{[l_n(x)]^2 [l_n^{(k)}(x)]^2}{x_f^j (k+1) l_n(x) + c_j (x)} \]

(3.4) \( W_j(x) = \frac{x^{k+1}l_n^j(x)}{\sum_{j=0}^{n+k} l_n^j(x)} \frac{[l_n(x)]^2 [l_n^{(k)}(x)]^2}{x_f^j (k+1) l_n(x) + c_j (x)} \]

(3.5) \( X_j(x) = \frac{x^{k+1} l_n^j(x)}{\sum_{j=0}^{n+k} l_n^j(x)} \frac{[l_n(x)]^3}{x_f^j (k+1) l_n(x)} \]

(3.6) \( C_j(x) = p_j(x) x^j \left[ L_n^j(x) \right]^2 \left[ L_n^{(k)}(x) \right]^2 \]

\( + x^k \left[ L_n^j(x) \right]^2 \left[ L_n^{(k)}(x) \right]^2 \frac{c_j^* - \frac{l_n^{(k)}(x) p_j(x) + q_j(x) l_n^{(k-j)}(x)}{x^{k-j}}} \]

(3.7) \( C_j(x) = \frac{1}{k + l - 1} \frac{1}{x^{k} L_n^{(k-j)}(x)} \frac{[l_n(x)]^3}{x_f^j (k+1) l_n(x)} \]

where \( p_j(x) \) and \( q_j(x) \) are polynomials of degree at most \( k-j-1 \) and \( c_j^* \) is an arbitrary constant.

**Theorem 3.2** Let the interpolatory function \( f: \mathcal{R} \rightarrow \mathcal{R} \) be continuously differentiable such that,

\( C(m) = |f(x)|: f \text{ is continuous in } [0, \infty), f(x) = 0(x^m) \text{ as } x \rightarrow \infty; \)

where \( m \geq 0 \) is an integer, then for every \( f \in C(m) \) and \( k \geq 0 \)

(3.8) \( R_n(x) = \sum_{j=0}^{n+k} a_j^{(n)} U_j(x) + \sum_{j=1}^{n+k} b_j^{(n)} V_j(x) + \sum_{j=1}^{n+k} c_j^{(n)} W_j(x) + \sum_{j=0}^{n+k} d_j^{(n)} C_j(x) \)

satisfies the relations

(3.9) \( \rho(x)|R_n(x) - f(x)| = O \left( n^{-\frac{k+1}{2}} \right) \omega \left( f, \frac{\log n}{\sqrt{n}} \right) \), for \( 0 \leq x \leq \Omega \)

(3.10) \( \rho(x)|R_n(x) - f(x)| = O \left( n^{-\frac{k+1}{2}} \right) \omega \left( f, \frac{\log n}{\sqrt{n}} \right) \), for \( \Omega^{-1} \leq x \leq \Omega \)

where \( \omega \) is the modulus of continuity.

**IV. PROOF OF THEOREM**

Let \( U_j(x), V_j(x), W_j(x), X_j(x) \) and \( C_j(x) \) are polynomials of degree \( \leq 4n+k \) satisfying conditions (4.1), (4.2), (4.3), (4.4) and (4.5) respectively.

\[
\begin{cases}
U_j(x_i) = 0 & \text{for } i \neq j, \\
U_j(x_i) = 1 & \text{for } i = j, \\
U_j(y_i) = 0 & \text{where } i = 1(1)n \text{ and } l = 0,1,...,k.
\end{cases}
\]

\[
\begin{cases}
V_j(x_i) = 0 & \text{for } i \neq j, \\
V_j(x_i) = 1 & \text{for } i = j, \\
V_j(y_i) = 0 & \text{where } i = 1(1)n \text{ and } l = 0,1,...,k.
\end{cases}
\]
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\begin{align}
W_j(x_i) = 0, \quad W_j'(x_i) = 0, \quad [\rho(x)W_j(x)]''_{x=x_i} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \\
and \\
W_j(y_l) = 0, \quad W_j^{(0)}(0) = 0,
\end{align}

\begin{align}
X_j(x_i) = 0, \quad X_j'(x_i) = 0, \quad [\rho(x)X_j(x)]''_{x=x_i} = 0,
\end{align}

\begin{align}
C_k(x_i) = 0, \quad C_j'(x_i) = 0, \quad [\rho(x)C_j(x)]''_{x=x_i} = 0,
\end{align}

To determine $X_j(x)$ let

\begin{align}
X_j^*(x) = C_1 x^{k+1} l_j'(x)[L_n^{(k)}(x)]^3
\end{align}

where $C_1$ is arbitrary constants. $l_j'(x)$ is defined in (2.9). $X_j^*(x)$ is a polynomial of degree $\leq 4n+k$

By using (2.8) we determine

\begin{align}
C_1 = \frac{1}{y_{i+1}^l [l_n^{(k)}(y_l)]^3}
\end{align}

Hence we find the fundamental polynomial $X_j(x)$ of degree $\leq 4n+k$

To find fundamental polynomial $W_j(x)$ let

\begin{align}
W_j^*(x) = C_2 x^{k+1} [l_n^{(k)}(x)]^2 L_n^{(k-1)}(x) l_j(x)
\end{align}

where $C_2$ is an arbitrary constants, $l_j(x)$ is defined in (2.8). $W_j^*(x)$ is polynomial of degree $\leq 4n+k$ satisfying the conditions (4.3) by which we obtain

\begin{align}
C_2 = \frac{e^{\epsilon/j^2}}{x_j^{(k+1)} L_n^{(k-1)}(x_j) [l_n^{(k)}(x_j)]^2}
\end{align}

Hence we find the fundamental polynomial $W_j(x)$ of degree $\leq 4n+k$

To find fundamental polynomial $V_j(x)$ let

\begin{align}
V_j^*(x) = C_3 x^{k+1} [l_j(x)]^2 L_n^{(k-1)}(x) [L_n^{(k)}(x)]^2 L_n^{(k-1)}(x)
\end{align}

using conditions (4.2) finding the value of all the constants we obtain the fundamental polynomial $V_j(x)$ of degree $\leq 4n+k$

Again let

\begin{align}
U_j^*(x) = x^{k+1} [l_j(x)]^3 L_n^{(k-1)}(x) C_5 + C_6 (x - x_j) + C_7 x^{k+1} l_j(x) [L_n^{(k)}(x)]^2
\end{align}

where $C_5, C_6$ and $C_7$ are arbitrary constants. $l_j(x)$ is defined in (2.8). $U_j^*(x)$ is polynomial of degree $\leq 4n+k$ satisfying the conditions (4.1) by which we obtain the fundamental polynomial $U_j(x)$ of degree $\leq 4n+k$.

To find $C_j(x)$, we assume $C_j(x)$ for fixed $j \in \{0, 1, \ldots, k-1\}$

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\[
(4.13) \quad C_j^*(\alpha) = p_j^*(\alpha)x^j[L_n^k(\alpha)]^2[\] \right]^{(k-1)}(\alpha) + x^k L_n^{(k-1)}(\alpha) [L_n^k(\alpha)]^2 g_n^*(\alpha)
\]

where \(p_j^*(\alpha)\) and \(g_n^*(\alpha)\) are polynomials of degree \((k-j)\) and \(n\) respectively. Now it is clear that \(C_j^*(0) = 0 \) for \((l = 0, \ldots, j - 1)\) and since \(L_n^{(k-1)}(\alpha) = 0\) and \(L_n^{(k-1)}(\alpha) = 0\) we get \(C_j(x) = 0\) and \(C_j(y_i) = 0\) for \(1 \leq i \leq n\). The coefficient of the polynomial \(p_j^*(\alpha)\) are calculated by

\[
C_j^*(0) = \frac{d^l}{{dx}^l}[p_j^*(\alpha)x^j[L_n^k(\alpha)]^2[L_n^{(k-1)}(\alpha)]^2]_{x=0} = \delta_{ij}
\]

\((l = j, \ldots, k - 1)\)

Now using the condition \([\rho(\alpha) C_j^*(\alpha)]_{x=\alpha} = 0\), we get

\[
g_n^*(\alpha) = -\frac{L_n^{(k-1)}(\alpha)p_j^*(\alpha)[g_n^*(\alpha)]}{x^k-j}
\]

which implies \(g_n^*(\alpha)\) as follows

\[
C_j^*(0) = \frac{d^l}{{dx}^l}[p_j^*(\alpha)x^j[L_n^k(\alpha)]^2[L_n^{(k-1)}(\alpha)]^2]_{x=0} = \delta_{ij}
\]

\((l = j, \ldots, k - 1)\)

Using \((4.13)\) and \((4.16)\) we obtain \(C_j^*(\alpha)\) of degree \(\leq 4n+k\) satisfying the conditions (4.5).

V. ESTIMATION OF THE FUNDAMENTAL POLYNOMIALS

**Lemma 5.1** Let the fundamental polynomial \(X_j(x)\), for \(j = 1,2,\ldots, n\) be given by \((3.5)\), then we have

\[
(5.1) \quad \sum_{j=1}^{n} e^{x_j/2} x_j^{-\frac{k-1}{2}} y_j^{-\frac{1}{2}} |X_j(x)| = O \left( n^\frac{k-1}{2} \right), \quad \text{for } 0 \leq x \leq cn^{-1}
\]

\[
(5.2) \quad \sum_{j=1}^{n} e^{x_j/2} x_j^{-\frac{k-1}{2}} y_j^{-\frac{1}{2}} |X_j(x)| = O \left( n^\frac{k-1}{2} \right), \quad \text{for } cn^{-1} \leq x \leq \Omega
\]

where \(X_j(x)\) is given in equation \((3.5)\).

**Proof:** From \((3.5)\) we have

\[
(5.3) \quad \sum_{j=1}^{n} e^{x_j/2} x_j^{-\frac{k-1}{2}} y_j^{-\frac{1}{2}} |X_j(x)| \leq \sum_{j=1}^{n} \left[ \frac{x^{k+1}[X_j^2(x)]}{x^{k+1}[X_j^2(y_j)]} \right]^{\frac{1}{2}}
\]

By equations \((2.13)\), \((2.16)\) and \((2.17)\) we yield the result.

**Lemma 5.2** Let the fundamental polynomial \(W_j(x)\), for \(j = 1,2,\ldots, n\) be given by \((3.4)\), then we have

\[
(5.4) \quad \sum_{j=1}^{n} |W_j(x)| = O \left( n^\frac{k-1}{2} \right), \quad \text{for } 0 \leq x \leq cn^{-1}
\]

\[
(5.5) \quad \sum_{j=1}^{n} |W_j(x)| = O \left( n^\frac{k-1}{2} \right), \quad \text{for } cn^{-1} \leq x \leq \Omega
\]

where \(W_j(x)\) is given in equation \((3.4)\).

**Proof:** From \((3.4)\) we have

\[
W_j(x) \leq \left[ \frac{e^{x_j/2} x_j^{-\frac{k-1}{2}} y_j^{-\frac{1}{2}} |X_j(x)|}{x_j^{k+1}[X_j^2(y_j)]} \right]^{\frac{1}{2}}
\]

\[
(5.6) \quad \sum_{j=1}^{n} |W_j(x)| \leq \sum_{j=1}^{n} \left[ \frac{e^{x_j/2} x_j^{-\frac{k-1}{2}} y_j^{-\frac{1}{2}} |X_j(x)|}{x_j^{k+1}[X_j^2(y_j)]} \right]^{\frac{1}{2}}
\]

Thus by using \((2.13)\) and \((2.17)\), \((3.4)\) and \((3.5)\) follows.

**Lemma 5.3** Let the fundamental polynomial \(V_j(x)\), for \(j = 1,2,\ldots, n\) be given by \((3.3)\), then we have

\[
(5.6) \quad \sum_{j=1}^{n} e^{x_j/2} x_j^{-\frac{k-1}{2}} |V_j(x)| = O \left( n^\frac{k-1}{2} \right), \quad \text{for } 0 \leq x \leq cn^{-1}
\]

\[
(5.7) \quad \sum_{j=1}^{n} e^{x_j/2} x_j^{-\frac{k-1}{2}} |V_j(x)| = O \left( n^\frac{k-1}{2} \right), \quad \text{for } cn^{-1} \leq x \leq \Omega
\]

where \(V_j(x)\) is given in \((3.3)\).

**Proof:** From \((3.3)\) we have

\[
|V_j(x)| \leq \left[ \frac{x^{k+1}[X_j^2(x)]}{x^{k+1}[X_j^2(y_j)]} \right]^{\frac{1}{2}}
\]

\[
+ \left[ \frac{x^{k+1}[X_j^2(x)]}{x^{k+1}[X_j^2(y_j)]} \right]^{\frac{1}{2}}
\]

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\[
\frac{e^{x_j/2} x_j^{-k-1/2} [V_j(x)]}{x_j^{k+1} ||f_n^{(k-1)}(x)||_{L_p^n(x)}} \\
\leq \sum_{j=1}^{n-1} \frac{e^{x_j/2} x_j^{-k-1/2} [f_n^{(k-1)}(x)] ||f_n^{(k-1)}(x)||_{L_p^n(x)}}{x_j^{k+1} ||f_n^{(k-1)}(x)||_{L_p^n(x)}} \\
+ \sum_{j=1}^{n} e^{x_j/2} x_j^{-k-1/2} [f_n^{(k-1)}(x)] ||f_n^{(k-1)}(x)||_{L_p^n(x)}^{2} \int_{x_j}^{x_{j+1}} \frac{e^{x_j/2} x_j^{-k-1/2} [f_n^{(k-1)}(x)] ||f_n^{(k-1)}(x)||_{L_p^n(x)}^{2}}{x_j^{k+1} ||f_n^{(k-1)}(x)||_{L_p^n(x)}^{2}} x_j(x_j^2) \\
= \zeta_{6.1} + \zeta_{6.2}
\]

Owing to (2.13) and (2.16), we get the result.

**Lemma 5.4** Let the fundamental polynomial \(U_j(x)\), for \(j = 1, 2, \ldots, n\) be given by equation (6.2), then we have

\[
\sum_{j=1}^{n} e^{x_j/2} x_j^{-k-1/2} [U_j(x)] = 0 \left(\begin{array}{c}
\frac{k-1}{2} \\
\frac{3-k}{2}
\end{array}\right), \quad \text{for } 0 \leq x \leq c n^{-1}
\]

\[
\sum_{j=1}^{n} e^{x_j/2} x_j^{-k-1/2} [U_j(x)] = 0 \left(\begin{array}{c}
\frac{k-1}{2} \\
\frac{3-k}{2}
\end{array}\right), \quad \text{for } c n^{-1} \leq x \leq \Omega
\]

Where \(U_j(x)\) is given in (3.2)

**Proof:** From (3.2) we have

\[
||U_j(x)|| \leq \frac{|x|^{k+1}|[f_j(x)]^{3}||f_{n-1}^{(1)}(x)|}{x_j^{k+1}||f_{n-1}^{(1)}(x)||} \\
+ \frac{|x|^{k+1}[[f_j(x)]^{3}||f_{n-1}^{(1)}(x)||]}{2 x_j^{k+1}||f_{n-1}^{(1)}(x)||}
\]

\[
\sum_{j=1}^{n} e^{x_j/2} x_j^{-k-1/2} [U(x)] \leq \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-k-1/2} [f_j(x)]^{3}||f_{n-1}^{(1)}(x)||}{x_j^{k+1}||f_{n-1}^{(1)}(x)||} \\
+ \sum_{j=1}^{n} e^{x_j/2} x_j^{-k-1/2} [f_j(x)]^{3}||f_{n-1}^{(1)}(x)||^{2} x_j(x_j^2) \\
= \zeta_{6.1} + \zeta_{6.2} + \zeta_{6.3}
\]

Where

\[
\zeta_{6.1} = \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-k-1/2} [f_j(x)]^{3}||f_{n-1}^{(1)}(x)||}{x_j^{k+1}||f_{n-1}^{(1)}(x)||} \\
\zeta_{6.2} = \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-k-1/2} [f_j(x)]^{3}||f_{n-1}^{(1)}(x)||^{2} x_j(x_j^2)}{2 x_j^{k+1}||f_{n-1}^{(1)}(x)||} \\
\zeta_{6.3} = \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-k-1/2} [f_j(x)]^{3}||f_{n-1}^{(1)}(x)||^{2} x_j(x_j^2)}{2 x_j^{k+1}||f_{n-1}^{(1)}(x)||}
\]

Thus by using (2.13) and (2.17), we yield the result.

Now we state our main theorem in § 6.

**VI. PROOF OF MAIN THEOREM 3.2**

Since \(R_n(x)\) given by equation (6.1) is exact for all polynomial \(Q_n(x)\) of degree \(\leq 4n+k\), we have

\[
Q_n(x) = \sum_{j=1}^{n} Q_n(x_j) U_j(x) + \sum_{j=1}^{n} Q_n(x_j)|W_j(x)| + \sum_{j=1}^{n} \rho(x) Q_n(x_j) + \sum_{j=1}^{n} \rho(x) Q_n(x_j) x_j + \sum_{j=1}^{n} \rho(x) Q_n(x_j) x_j \]

From equation (6.2.1) and (6.4.1) we get

\[
\rho(x)f(x) - R_n(x) \leq \rho(x)f(x) - Q_n(x) + \rho(x) Q_n(x) - R_n(x)
\]

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Thus (6.2), and Lemmas 5.1, 5.2, 5.3 and 5.4 completes the proof of the theorem.

REFERENCES


