IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 07, Issue 03 (March. 2017), ||V1|| PP 11-27

Initial and Boundary Value Problems Involving the Inhomogeneous Generalized Airy's Equations

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Abstract: Initial and boundary value problems of the inhomogeneous Airy's and generalized Airy's differential equations are considered in this work. General solutions are expressed in terms of the Nield-Kuznetsov functions of the first and second kinds, and are computed when the forcing function is a constant or a variable function of the independent variable.

Keywords: *Generalized Airy's inhomogeneous equations, Nield-Kuznetsov functions.*

I. INTRODUCTION

In their pioneering work on flow through porous layers, as governed by Brinkman's equation, Nield and Kuznetsov [1] introduced the concept of transition layer, defined here as a porous layer of variable permeability imbedded between two constant permeability porous layers, or one that is bounded by a constant permeability on one side and free-space on the other. They modelled the flow through the transition layer using Brinkman's equation with variable permeability using a permeability function that ingeniously reduced the governing equation to an inhomogeneous Airy's differential equation with constant forcing function. Solution to the flow problem was then attained in terms of an integral function, $Ni(x)$, they introduced and defined in terms of Airy's functions of the first and second kind, and used asymptotic series representations of Airy's functions of the first and second kind to evaluate the $Ni(x)$ function.

The $Ni(x)$ function was subsequently studied extensively by Hamdan and Kamel [2] who documented its properties and extended its introduction to the integral function $Ki(x)$ that arises in the solution of the inhomogeneous Airy's equation with variable forcing function. The functions $Ni(x)$ and $Ki(x)$ have since been recognized as the Nield-Kuznetsov functions. Representations of these functions in terms of both asymptotic and ascending series have been obtained by Alzahrani *et al*. [3, 4], who provided a comparison of the solutions to flow through the transition layer using both of these series representations.

Abu Zaytoon *et al*. [5] approached the problem of flow through transition layer by modelling its permeability using a function that reduced the governing Brinkman's equation to the generalized inhomogeneous Airy's differential equation, of index *n*, with constant forcing function, and recovered the solution obtained by Nield and Kuznetsov [1] by choosing $n = 1$. We point out here that the generalized homogeneous Airy's differential equation has been extensively studied by Swanson and Headley [6]. In case of the generalized inhomogeneous Airy's equation with constant forcing function, Abu Zaytoon *et al*. [5] expressed its general solution with the help of a generalized form of the Nield-Kuznetsov $Ni(x)$ function, referred to as the generalized $Ni(x)$ function (or $N_n(x)$ function), who also provided its series representation using series representations of the generalized Airy's functions discussed in [6]. For the case of generalized, inhomogeneous Airy's equation with variable forcing function, Alzahrani *et al*. [7] have recently provided a general solution in terms of a generalized form of the $Ki(x)$ function, which they denoted by $K_n(x)$, and derived appropriate series representations for its evaluation. Due to the arize of various forms of the Nield-Kuznetsov functions, Alzahrani *et al*. [7] adopted the following acronyms:

(i) $Ni(x)$ and $Ki(x)$ are termed the Standard Nield-Kuznetsov functions of the first- and second-kind, respectively. They arise in the solution to the inhomogeneous Airy's equation with constant and variable forcing functions, respectively.

(ii) $N_n(x)$ and $K_n(x)$ are termed the generalized Nield-Kuznetsov functions of the first- and second-kind, respectively. They arise in the solution to the inhomogeneous generalized Airy's equation of index *n* with constant and variable forcing functions, respectively.

The importance of the Airy's and generalized Airy's equations, and the above functions, in the modelling and solution of flow through porous layers, and their potential applications to other problems in mathematical physics motivates the current work in which we provide computations and analysis of initial and boundary value problems involving the inhomogeneous Airy's and generalized Airy's equations. Problem statements and solutions are provided for both constant and variable forcing functions in order to study the effects of the forcing functions on the solutions obtained.

II. PROBLEM FORMULATION

Required to solve the generalized Airy's inhomogeneous ordinary differential equation (ODE):

$$u'' - y^n u = f(y) \qquad \ldots(1)$$

subject to the initial conditions (I.C.)

$$u(0) = \alpha \text{ and } u'(0) = \beta \qquad \ldots(2)$$

where α and β are known constants, or subject to the boundary conditions (B.C.)

$$u(a_1) = b_1 \text{ and } u(a_2) = b_2 \qquad \ldots(3)$$

where a_1, a_2, b_1 and b_2 are real numbers.

In equation (1), *n* is a positive integer, prime notation denotes ordinary differentiation with respect to the independent variable *y*, and $f(y)$ is the forcing function.

General solution to equation (1) is given by, [7]:

$$u_n = c_{1n} A_n(y) + c_{2n} B_n(y) - \frac{\pi}{2\sqrt{m}\sin(m\pi)} K_n(y) \qquad \ldots(4)$$

where $m = \frac{1}{n+2}$, c_{1n}, c_{2n} are arbitrary constants, $A_n(y)$ and $B_n(y)$ are the generalized Airy's functions of the first- and second-kind, respectively, [6], and $K_n(y)$ is the generalized Nield-Kuznetsov function of the second-kind, defined by, [7]

$$K_n(y) = -[A_n(y)\int_0^y F(t)B_n'(t)dt - B_n(y)\int_0^y F(t)A_n'(t)dt] \qquad \ldots(5)$$

with first derivative given by

$$K_n'(y) = -\left[A_n'(y)\int_0^y F(t)B_n'(t)dt - B_n'(y)\int_0^y F(t)A_n'(t)dt + \frac{2\sqrt{m}\sin(m\pi)}{\pi}F(y)\right] \qquad \ldots(6)$$

wherein $F'(y) = f(y)$. When *n=1*, equation (1) reduces to the well-known Airy's ODE whose general solution is given by, [7]:

$$u = c_1 Ai(y) + c_2 Bi(y) - \pi Ki(y) \qquad \ldots(7)$$

where $Ai(y)$ and $Bi(y)$ are Airy's functions of the first- and second-kind, c_1 and c_2 are arbitrary constants, and $Ki(y)$ is the standard Nield-Kuznetsov function of the second-kind defined by, [7]:

$$Ki(y) = -[Ai(y)\int_0^y F(t)B'i(t)dt - Bi(y)\int_0^y F(t)A'i(t)dt] \qquad \ldots(8)$$

with first derivative given by

$$K'i(y) = -\left[A'i(y)\int_0^y F(t)B'i(t)dt - B'i(y)\int_0^y F(t)A'i(t)dt + \frac{1}{\pi}F(y)\right]. \qquad \ldots(9)$$

When the forcing function is a constant, say $f(y) = \kappa$, general solutions (4) and (7) reduce, respectively, to

$$u_n = c_{1n}A_n(y) + c_{2n}B_n(y) - \frac{\kappa\pi}{2\sqrt{m}\sin(m\pi)}N_n(y) \quad \ldots(10)$$

and

$$u = c_1 Ai(y) + c_2 Bi(y) - \pi\kappa Ni(y) \quad \ldots(11)$$

where $N_n(y)$ is the generalized Nield-Kuznetsov function of the first kind and $Ni(y)$ is the standard Nield-Kuznetsov function of the first-kind defined, respectively, by, [7]:

$$N_n(y) = A_n(y)\int_0^y B_n(t)dt - B_n(y)\int_0^y A_n(t)dt \quad \ldots(12)$$

and

$$Ni(y) = Ai(y)\int_0^y Bi(t)dt - Bi(y)\int_0^y Ai(t)dt\ . \quad \ldots(13)$$

First derivatives of $N_n(y)$ and $Ni(y)$ are given, respectively, by

$$N_n'(y) = A_n'(y)\int_0^y B_n(t)dt - B_n'(y)\int_0^y A_n(t)dt \quad \ldots(14)$$

$$N'i(y) = A'i(y)\int_0^y Bi(t)dt - B'i(y)\int_0^y Ai(t)dt\ . \quad \ldots(15)$$

In order to obtain complete solutions to the initial and boundary value problems, general solutions (4) must satisfy condition (2) for initial value problem and condition (3) for boundary value problem. This leads to determination of the arbitrary constants appearing in (4). In what follows, the arbitrary constants are determined for cases of constant and variable forcing functions, for the initial and boundary value problems.

III. SOLUTION TO THE INITIAL VALUE PROBLEMS (IVP)

Using initial condition (2) in the general solution (4) results in the following values for the arbitrary constants c_{1n} and c_{2n}:

$$c_{1n} = \frac{\alpha B_n'(0) - B_n(0)\left(\beta + \pi\dfrac{\pi}{2\sqrt{m}\sin(m\pi)}K_n'(0)\right)}{A_n(0)B_n'(0) - A_n'(0)B_n(0)} \quad \ldots(16)$$

$$c_{2n} = \frac{-\alpha A_n'(0) + A_n(0)\left(\beta + \pi\dfrac{\pi}{2\sqrt{m}\sin(m\pi)}K_n'(0)\right)}{A_n(0)B_n'(0) - A_n'(0)B_n(0)}\ . \quad \ldots(17)$$

Upon substituting (16) and (17) in (4), solution to the IVP is completely determined.

When the forcing function is of the form $f(y) = \kappa$, where κ is a specified constant, c_{1n} and c_{2n} take the following forms:

$$c_{1n} = \frac{\alpha B_n'(0) - \beta B_n(0)}{A_n(0)B_n'(0) - A_n'(0)B_n(0)} \quad \ldots(18)$$

$$c_{2n} = \frac{-\alpha A_n'(0) + \beta A_n(0)}{A_n(0)B_n'(0) - A_n'(0)B_n(0)}\ . \quad \ldots(19)$$

Upon substituting (18) and (19) in (7), solution to the IVP is completely determined.

IV. SOLUTION TO THE BOUNDARY VALUE PROBLEMS (BVP)

Using boundary condition (3) in the general solution (4) results in the following values for the arbitrary constants c_{1n} and c_{2n}:

$$c_{1n} = \frac{b_1 B_n(a_2) - b_2 B_n(a_1) + \dfrac{\pi}{2\sqrt{m}\sin(m\pi)}[K_n(a_1)B_n(a_2) - K_n(a_2)B_n(a_1)]}{A_n(a_1)B_n(a_2) - A_n(a_2)B_n(a_1)} \quad \ldots(20)$$

$$c_{2n} = \frac{b_1 A_n(a_2) - b_2 A_n(a_1) + \dfrac{\pi}{2\sqrt{m}\sin(m\pi)}[K_n(a_1)A_n(a_2) - K_n(a_2)A_n(a_1)]}{B_n(a_1)A_n(a_2) - B_n(a_2)A_n(a_1)}. \quad \ldots(21)$$

Upon substituting (20) and (21) in (4), solution to the BVP is completely determined.

When the forcing function is $f(y) = \kappa$, where κ is a specified constant, c_{1n} and c_{2n} take the following forms:

$$c_{1n} = \frac{b_1 B_n(a_2) - b_2 B_n(a_1) + \dfrac{\kappa\pi}{2\sqrt{m}\sin(m\pi)}[N_n(a_1)B_n(a_2) - N_n(a_2)B_n(a_1)]}{A_n(a_1)B_n(a_2) - A_n(a_2)B_n(a_1)} \quad \ldots(22)$$

$$c_{2n} = \frac{b_1 A_n(a_2) - b_2 A_n(a_1) + \dfrac{\kappa\pi}{2\sqrt{m}\sin(m\pi)}[N_n(a_1)A_n(a_2) - N_n(a_2)A_n(a_1)]}{B_n(a_1)A_n(a_2) - B_n(a_2)A_n(a_1)}. \quad \ldots(23)$$

Upon substituting (22) and (23) in (7), solution to the BVP is completely determined.

V. COMPUTATIONS OF SOLUTIONS TO IVP AND BVP

V.1. Series Expressions for the Generalized Functions

Determination of values of the arbitrary constants, appearing in the IVP and BVP, and the evaluation of their solutions at particular values of the independent variable, *y*, necessitates evaluations of the standard and generalized Nield-Kuznetsov functions at the given values of *y*.

At the outset, the following values of the Nield-Kuznetsov functions at $y = 0$ have been obtained from their definitions, equations (5), (6), (8), (9), and (12)-(15), and used in deriving expressions (16)-(23) of the arbitrary constants:

$$N_n(0) = N_n'(0) = K_n(0) = 0,\ K'i(0) = -\frac{1}{\pi}F(0);\ K_n'(0) = -\frac{2\sqrt{m}\sin(m\pi)}{\pi}F(0). \quad \ldots(24)$$

Computations of the Airy's, Generalized Airy's, the standard and generalized Nield-Kuznetsov functions and their derivatives at any value of the independent variable *y*, are discussed in what follows.

The generalized Airy's functions have been shown to have the following power series expansions [6]:

$$A_n(y) = \alpha_n g_{n1}(y) - \beta_n g_{n2}(y) \quad \ldots(25)$$

$$B_n(y) = [\alpha_n g_{n1}(y) + \beta_n g_{n2}(y)] / \sqrt{m} \quad \ldots(26)$$

$$\alpha_n = (m^{1-m}) / \Gamma(1-m) \quad \ldots(27)$$

$$\beta_n = (m^m) / \Gamma(m) \quad \ldots(28)$$

$$g_{n1}(y) = 1 + \sum_{j=1}^{\infty} m^{2j} \prod_{p=1}^{j} \frac{y^{j(n+2)}}{p(p-m)} \quad \ldots(29)$$

$$g_{n2}(y) = y[1 + \sum_{j=1}^{\infty} m^{2j} \prod_{p=1}^{j} \frac{y^{j(n+2)}}{p(p+m)}]. \quad \ldots(30)$$

Equations (25)-(30) can evaluated at $y = 0$ to generate the values below for Airy's and generalized Airy's functions and their first derivatives, where the values of Airy's functions and first derivatives at $y = 0$ are obtained using n=1 and m=1/3, (cf. [2,5,7]):

$$Ai(0) = \frac{(1/3)^{2/3}}{\Gamma(2/3)};\ A'(0) = -\frac{(1/3)^{1/3}}{\Gamma(1/3)};\ Bi(0) = \frac{(1/3)^{1/6}}{\Gamma(2/3)};\ B'(0) = \frac{(3)^{1/6}}{\Gamma(1/3)} \quad \ldots(31)$$

$$A_n(0) = \frac{(m)^{1-m}}{\Gamma(1-m)};\ A_n'(0) = -\frac{(m)^m}{\Gamma(m)};\ B_n(0) = \frac{(m)^{1/2-m}}{\Gamma(1-m)};\ B_n'(0) = \frac{(m)^{m-1/2}}{\Gamma(m)} \quad \ldots(32)$$

The generalized Airy's functions, given in equations (25) and (26), above, have been used by Alzahrani *et al.* [3,7] to derive the following series expressions for the generalized Nield-Kuznetsov functions $N_n(y)$ and $K_n(y)$:

$$N_n(y) = \frac{2}{\sqrt{m}}\alpha_n\beta_n\left[g_{n1}(y)\int_0^y [g_{n2}(t)]\,dt - g_{n2}(y)\int_0^y [g_{n1}(t)]\,dt \right] \quad \ldots(33)$$

$$K_n(y) = \frac{-2}{\sqrt{m}}\alpha_n\beta_n\left[g_{n1}(y)\int_0^y [F(t)g'_{n2}(t)]\,dt - g_{n2}(y)\int_0^y [F(t)g'_{n1}(t)]\,dt \right]. \quad \ldots(34)$$

$$g'_{n1}(y) = \sum_{j=1}^{\infty} m^{2j-1} j^2 \left(y^{-1+j^2/m}\right)\prod_{p=1}^{j}\frac{1}{p(p-m)} \quad \ldots(35)$$

$$g'_{n2}(y) = 1 + \sum_{j=1}^{\infty} m^{2j}(1+\frac{j^2}{m})\left(y^{j^2/m}\right)\prod_{p=1}^{j}\frac{1}{p(p+m)}] \quad \ldots(36)$$

$$\int_0^y [g_{n1}(t)]\,dt = y + \sum_{j=1}^{\infty} m^{2j}\frac{1}{(1+j^2/m)}\left(y^{1+j^2/m}\right)\prod_{p=1}^{j}\frac{1}{p(p-m)} \quad \ldots(37)$$

$$\int_0^y [g_{n2}(t)]\,dt = \frac{y^2}{2} + \sum_{j=1}^{\infty} m^{2j}\frac{1}{(2+j^2/m)}\left(y^{2+j^2/m}\right)\prod_{p=1}^{j}\frac{1}{p(p+m)}]. \quad \ldots(38)$$

Equations (24)-(38) are used to compute the generalized Airy's and generalized Nield-Kuznetsov functions appearing in the solutions to the IVP and BVP. Computations are illustrated in the following numerical experiment.

V.2. Numerical Experiment

For constant forcing function $f(y) = \kappa = 1/\pi$ and $f(y) = \kappa = -1/\pi$, and for variable forcing functions $f(y) = y$, $f(y) = y^2$ and $f(y) = \sin y$, suppose it is desired to solve equation (1) subject to the initial conditions $u(0) = 2, u'(0) = 1$ and subject to the boundary conditions $u(0) = 1, u(1) = 2$. Then, using equations (24)-(38), the integral functions appearing in expressions (16)-(23) for the arbitrary constants. Solutions (4) and (7) to IVP, and (10) and (11) to BVP, are then evaluated and plotted over the interval $0 \le y \le 1$ for various values of generalized Airy's parameter *n*, as discussed in the next section.

VI. RESULTS AND DISCUSSION

VI.1. Solution to Airy's Equation with Initial Values

When *n* = 1, equation (1) is the well-known Airy's differential equation. In this case, solutions (7) and (11) with initial conditions and with either constant or variable forcing functions result in the same values for the arbitrary constants, computed using expression (16)-(19): c_1 = 0.8848298434, c_2 = 2.741563801. Solutions (7) and (11) are illustrated graphically in **Fig. 1(a)** and **1(b)**. For constant forcing functions, **Fig. 1(a)** illustrates solution (11) and shows an exponential increase in $u(y)$ over the interval $0 \le y \le 1$ for both $f(y) = \mp 1/\pi$, with a sharper increase when $f(y) = 1/\pi$. It is noted that the solutions obtained here using the discussed procedure is an alternative method to the solutions obtained for the same problem using Scorer functions, [2].

Fig. 1(b) illustrates solution (7) for the variable forcing functions $f(y) = y$, $f(y) = y^2$ and $f(y) = \sin y$, and shows the relative positions of the exponentially increasing curves for the functions tested. Solution curves are close to each other due to the closeness of the values of the functions over the interval $0 \le y \le 1$. For larger values of *y*, it is expected that the solution curve for $f(y) = y^2$ will intersect the other two curves and grow exponentially faster, relative to the other two curves.

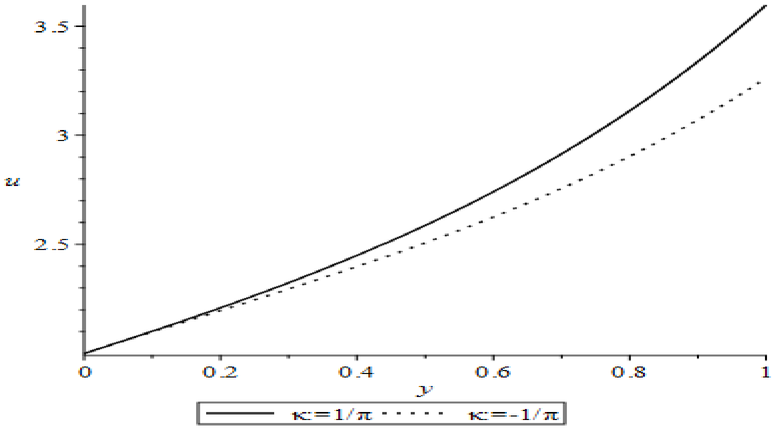

Fig. 1(a) Solutions to Airy's IVP with Constant Forcing Function $f(y) = \mp 1/\pi$

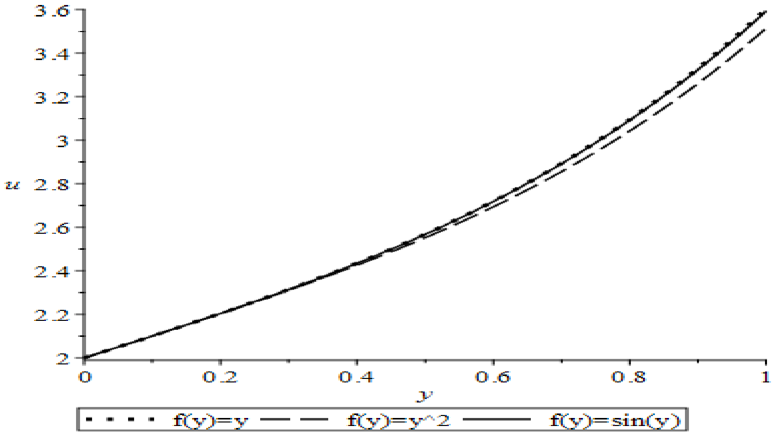

Fig. 1(b) Solutions to Airy's IVP with Variable Forcing Function $f(y)$

VI.2. Solution to Airy's Equation with Boundary Values

In this case, solutions (7) and (11) with boundary conditions and with either constant or variable forcing functions result in values for the arbitrary constants shown in **Table 1**. These have been computed using expressions (20)-(23).

Table 1. Values of arbitrary constants in the solution to BVP involving Airy's equation.

$f(y)$	c_1	c_2
$1/\pi$	0.2327830325	1.491812929
$-1/\pi$	-0.3626315184	1.835575680
y	0.2417608772	1.486629567
y^2	0.08697595842	1.575994682
$\sin y$	0.2270128925	1.495144321

Solutions (7) and (11) are illustrated graphically in **Fig. 2(a)** and **2(b)**. For constant forcing functions, **Fig. 2(a)** illustrates solution (11) and shows the relative positions of the solution curves, and how the solutions increase, over the interval $0 \le y \le 1$ for both $f(y) = \mp 1/\pi$. **Fig. 2(b)** illustrates solution (7) for the variable forcing functions $f(y) = y$, $f(y) = y^2$ and $f(y) = \sin y$, and shows the relative positions and closeness of the solution curves for the functions tested. Again, solution curves are close to each other due to the closeness of the values of the functions over the interval $0 \le y \le 1$. In both cases of constant or variable forcing functions, solutions $u(y)$ are higher for functions with lower values of y. In other words, over the interval $0 \le y \le 1$, $u(y)$ when $f(y) = 1/\pi$ is less than $u(y)$ when $f(y) = -1/\pi$. The same conclusion is holds for variable forcing functions.

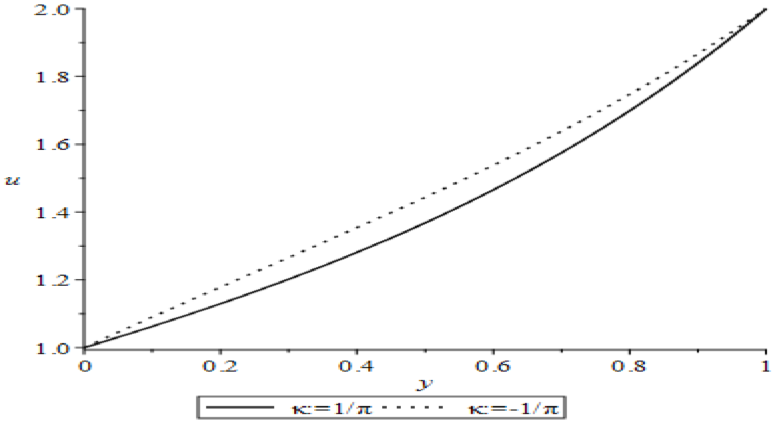

Fig. 2(a) Solutions to Airy's BVP with Constant Forcing Function $f(y) = \mp 1/\pi$

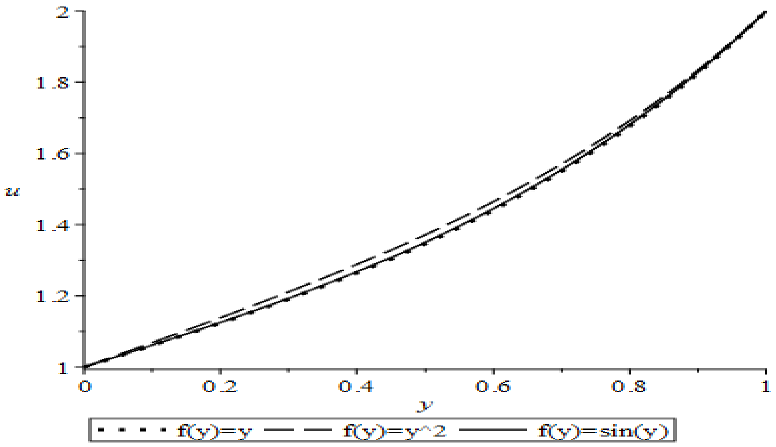

Fig. 2(b) Solutions to Airy's BVP with Variable Forcing Function $f(y)$

VI.3. Solution to Generalized Airy's Equation with Initial Values

In order to study the effects of increasing the generalized Airy parameter *n* on the solution to the inhomogeneous generalized Airy's equation with initial conditions, solution to equation (1) is evaluated for *n* = 1, 2, 3, 4, 5 and 10. For the initial value problem with constant forcing function, expressions (18) and (19) for the arbitrary constants c_{1n} and c_{2n} are evaluated for various values of *n* and shown in **Table 2**. Solutions (10) for u_n are evaluated and plotted for each of the constant functions $f(y) = \mp 1/\pi$ in **Fig. 3(a,b)** and **4(a,b)**. **Fig. 3(a,b)** illustrate u_n for $f(y) = 1/\pi$ and the various values of *n*, while **Fig. 4(a,b)** illustrate u_n for $f(y) = -1/\pi$ and the various values of *n*. For visual clarity, the figures group the cases of *n* = 1,2,3 in one graph and *n* = 4,5,10 in another graph. All of these graphs show the relative positions of the solution curves with increasing *n*, and demonstrate the decrease in u_n with increasing *n* over the interval $0 \le y \le 1$, with larger decrease as *y* increases. This pattern persists for both constant forcing functions considered.

Table 2. IVP Values of c_{1n} and c_{2n} for different values of *n* and constant or variable $f(y)$.

	$f(y) = \kappa = \mp \frac{1}{\pi}$
$n = 1$	$c_{1n} = 0.88482984; c_{2n} = 2.741563801$
$n = 2$	$c_{1n} = 0.9023084857;$ $c_{2n} = 3.014847595$

$n = 3$	$c_{1n} = 1.051988482$; $c_{2n} = 3.303169579$
$n = 4$	$c_{1n} = 1.272537560$; $c_{2n} = 3.582773032$
$n = 5$	$c_{1n} = 1.538574592$; $c_{2n} = 3.849602302$
$n = 10$	$c_{1n} = 3.224800671$; $c_{2n} = 5.014277165$

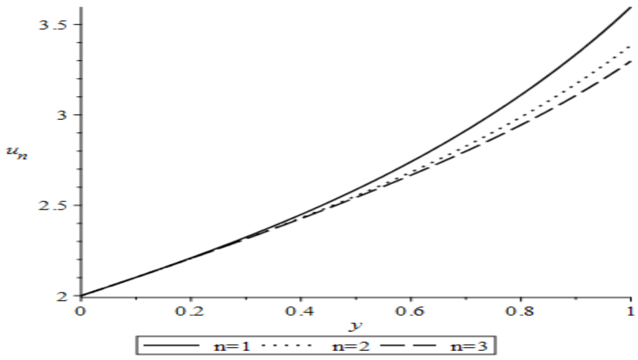

Fig. 3(a) Solutions to Generalized Airy's IVP with Constant Forcing Function $f(y) = 1/\pi$ and n = 1, 2 and 3.

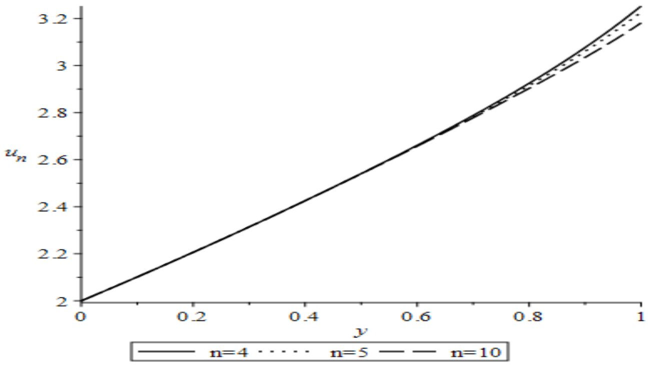

Fig. 3(b) Solutions to Generalized Airy's IVP with Constant Forcing Function $f(y) = 1/\pi$ and n = 4, 5 and 10.

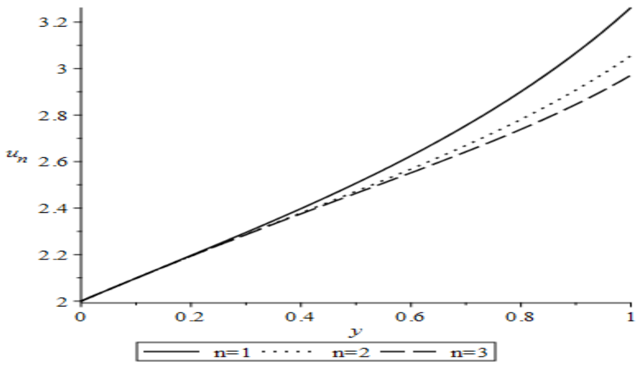

Fig. 4(a) Solutions to Generalized Airy's IVP with Constant Forcing Function $f(y) = -1/\pi$ and n = 1, 2, 3.

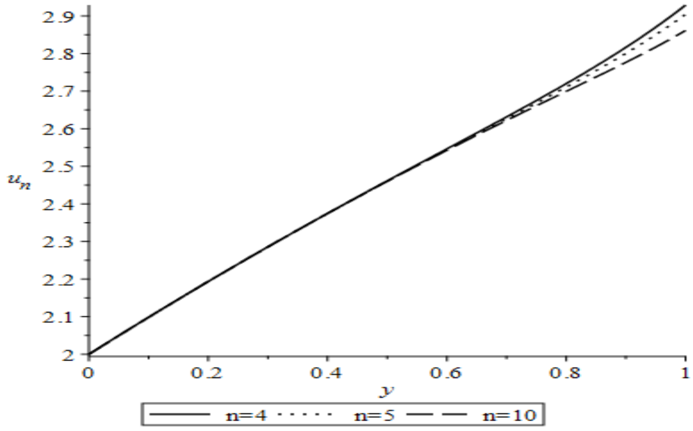

Fig. 4(b) Solutions to Generalized Airy's IVP with Constant Forcing Function $f(y) = -1/\pi$ and n = 4, 5, 10.

For the initial value problem with variable forcing functions, expressions (16) and (17) for the arbitrary constants c_{1n} and c_{2n} are evaluated for various values of n and shown in **Table 2** for each of the variable forcing functions considered. Solutions (4) for u_n are evaluated and plotted in **Fig. 5(a,b)**, **6(a,b), 7(a,b)** and **8(a,b)**.

Fig. 5(a) illustrates u_n for n = 1 (namely, solution to Airy's equation) and **Fig. 5(b)** illustrates u_n for n = 10, for various variable forcing functions. These figures demonstrate the similarity in qualitative behaviour of the solutions to Airy's and generalized Airy's equations for the different forcing functions tested.

Fig. 6(a,b), 7(a,b), 8(a,b) illustrate u_n for $f(y) = y$, $f(y) = y^2$ and $f(y) = \sin y$, respectively, and the various values of n. Again, for visual clarity, the figures group the cases of n = 1,2,3 in one graph and n = 4,5,10 in another graph. All of these graphs show the relative positions of the solution curves with increasing n, and demonstrate the decrease in u_n with increasing n over the interval $0 \le y \le 1$, with larger decrease as y increases. This pattern persists for all forcing functions considered.

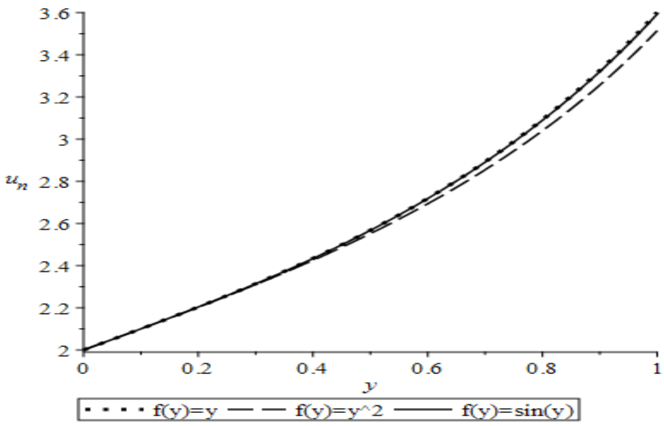

Fig. 5(a) Solutions to Airy's IVP, n = 1, with Various Variable Forcing Functions $f(y)$

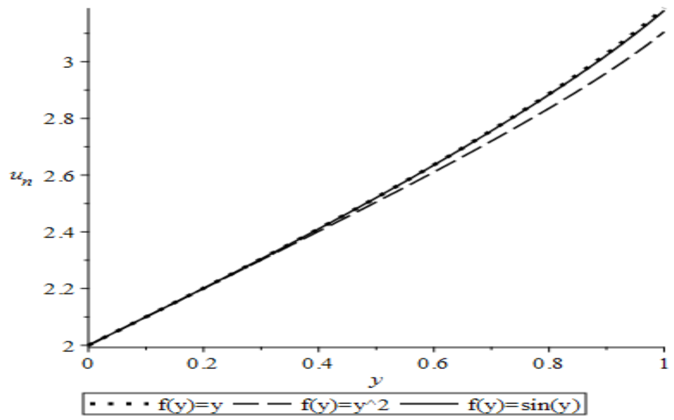

Fig. 5(b) Solutions to Generalized Airy's IVP with Various Variable Forcing Functions $f(y)$ and n = 10

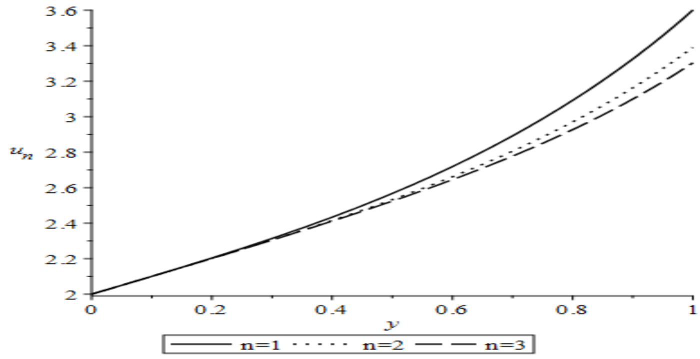

Fig. 6(a) Solutions to Generalized Airy's IVP with $f(y) = y$ and n = 1, 2, and 3

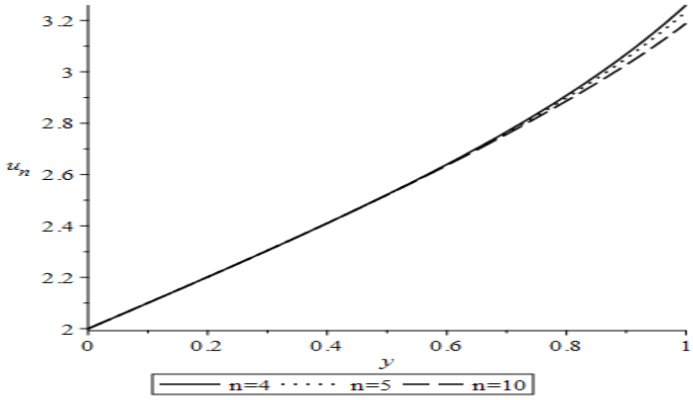

Fig. 6(b) Solutions to Generalized Airy's IVP with $f(y) = y$ and n = 4, 5, and 10

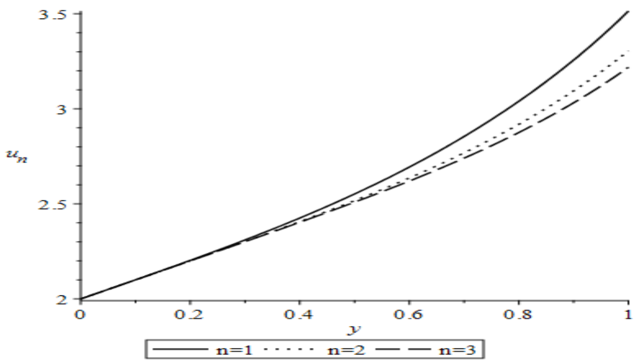

Fig. 7(a) Solutions to Generalized Airy's IVP with $f(y) = y^2$ and n = 1, 2, and 3

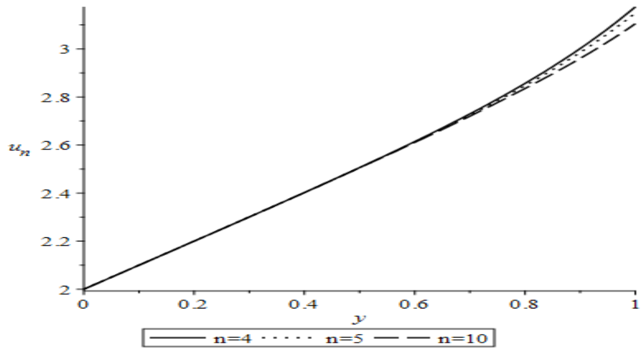

Fig. 7(b) Solutions to Generalized Airy's IVP with $f(y) = y^2$ and n = 4, 5, and 10

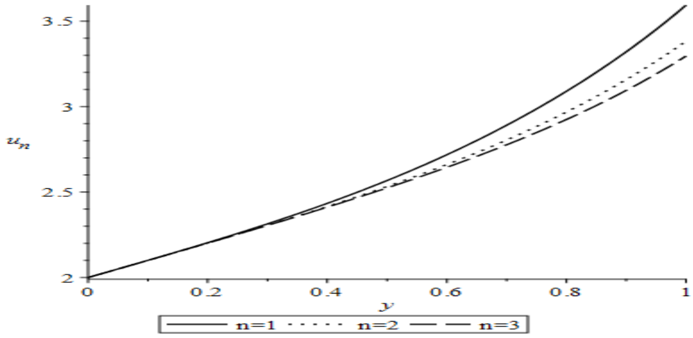

Fig. 8(a) Solutions to Generalized Airy's IVP with $f(y) = \sin y$ and n = 1, 2, and 3

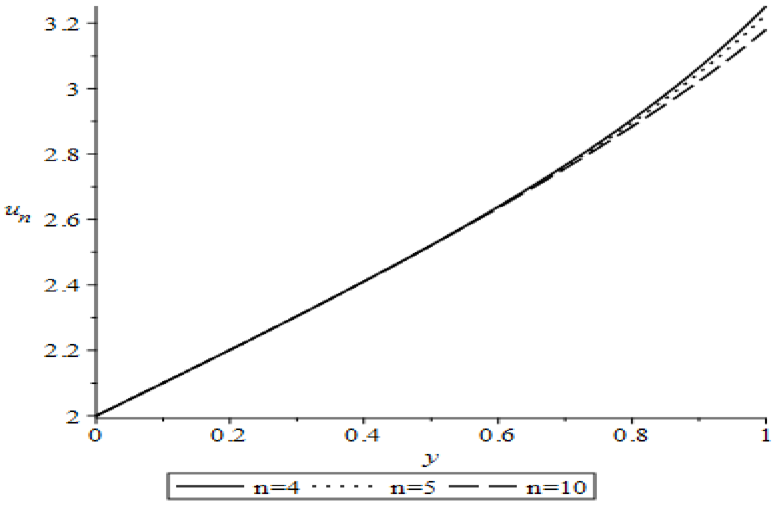

Fig. 8(b) Solutions to Generalized Airy's IVP with $f(y) = \sin y$ and n = 4, 5, and 10

VI.4. Solution to Generalized Airy's Equation with Boundary Values

In order to study the effects of increasing the generalized Airy parameter n on the solution to the inhomogeneous generalized Airy's equation with boundary conditions, solution to equation (1) is evaluated for n = 1, 2, 3, 4, 5 and 10. For the boundary value problem with constant forcing function, expressions (22) and (23) for the arbitrary constants c_{1n} and c_{2n} are evaluated for various values of n and shown in **Table 3**. Solution (10) for u_n is evaluated and plotted for the constant functions $f(y) = \mp 1/\pi$. **Fig. 9(a)** illustrates u_n for n = 1 (namely, solution to Airy's equation) and **Fig. 9(b)** illustrates u_n for n = 10, for the constant forcing functions $f(y) = \mp 1/\pi$. These figures demonstrate the similarity in qualitative behaviour of the solutions to Airy's and generalized Airy's equations. **Fig. 10(a,b)** and **11(a,b)** illustrate u_n for $f(y) = 1/\pi$

and $f(y) = -1/\pi$, respectively, for various values of *n*. Again, for visual clarity, the figures group the cases of *n* = 1,2,3 in one graph and *n* = 4,5,10 in another graph. All of these graphs show the relative positions of the solution curves with increasing *n*, and demonstrate the increase in u_n with increasing *n* over the interval $0 \le y \le 1$. This pattern persists for both constant forcing functions considered.

Table 3. BVP values of c_{1n} and c_{2n} for different values of *n* and constant $f(y)$.

	$f(y) = \kappa = \frac{1}{\pi}$	$f(y) = \kappa = -\frac{1}{\pi}$
$n = 1$	c_{1n} = 0.2326261442; c_{2n} = 1.491903508	c_{1n} = -0.3624746303; c_{2n} = 1.83548510
$n = 2$	c_{1n} = -0.09860910126; c_{2n} = 1.782305479	c_{1n} = -0.9013843510; c_{2n} = 2.183693094
$n = 3$	c_{1n} = -0.3002242867; c_{2n} = 2.02108094	c_{1n} = -1.298897435; c_{2n} = 2.467701154
$n = 4$	c_{1n} = -0.4351366251; c_{2n} = 2.228785943	c_{1n} = -1.622328325; c_{2n} = 2.713454925
$n = 5$	c_{1n} = -0.5290973414; c_{2n} = 2.415544415	c_{1n} = -1.899826909; c_{2n} = 2.933631495
$n = 10$	c_{1n} = -0.7008235147; c_{2n} = 3.174908784	c_{1n} = -2.950042667; c_{2n} = 3.824202427

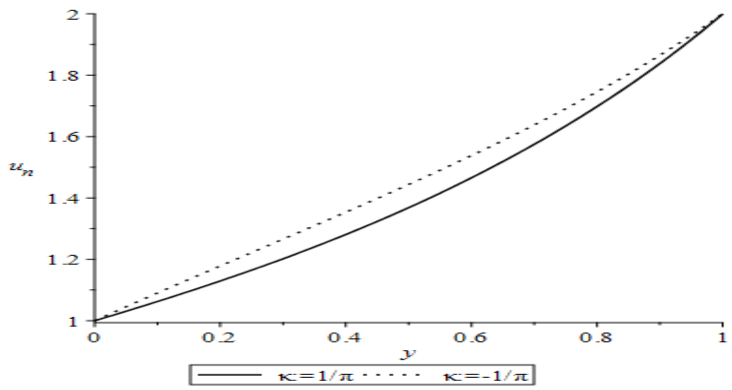

Fig. 9(a) Solutions to Airy's BVP with Constant Forcing Functions $f(y) = \kappa = \mp 1/\pi$

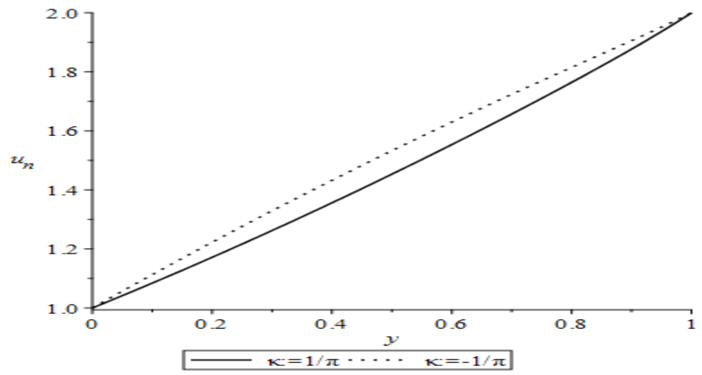

Fig. 9(b) Solutions to Generalized Airy's BVP with Constant Forcing Functions $f(y) = \kappa = \mp 1/\pi$; *n* = 10

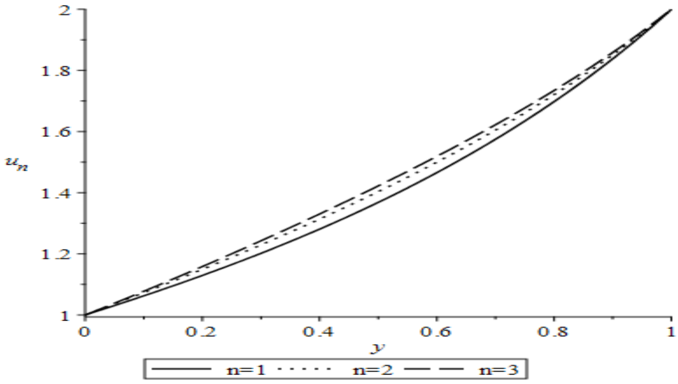

Fig. 10(a) Solutions to Generalized Airy's BVP with Constant Forcing Function $f(y) = \kappa = 1/\pi$ and n = 1, 2, and 3

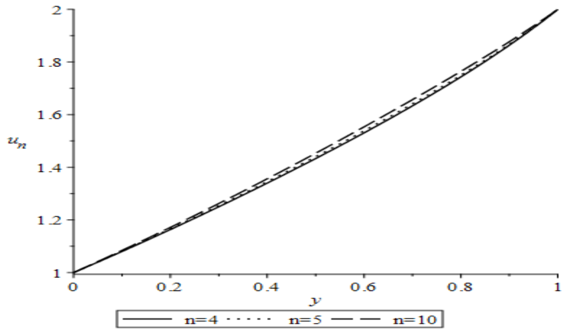

Fig. 10(b) Solutions to Generalized Airy's BVP with Constant Forcing Function $f(y) = \kappa = 1/\pi$ and n = 4, 5, and 10

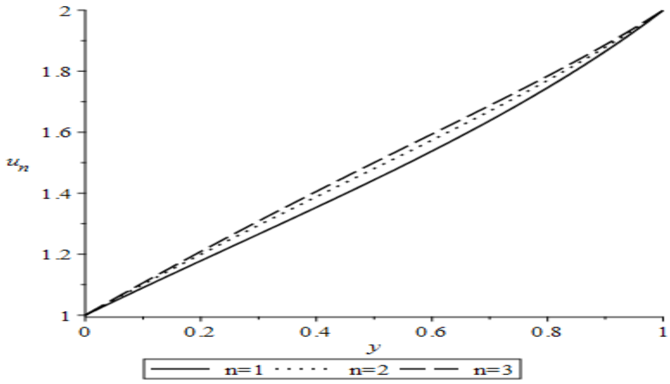

Fig. 11(a) Solutions to Generalized Airy's BVP with Constant Forcing Function $f(y) = \kappa = -1/\pi$ and n = 1, 2, and 3

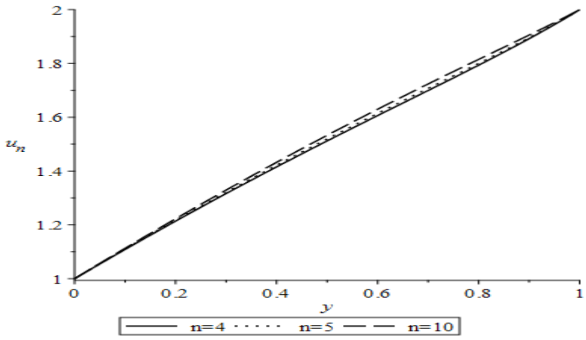

Fig. 11(b) Solutions to Generalized Airy's BVP with Constant Forcing Function $f(y) = \kappa = -1/\pi$ and n = 4, 5, and 10

For the boundary value problem with variable forcing functions, expressions (20) and (21) for the arbitrary constants c_{1n} and c_{2n} are evaluated for various values of *n* and shown in **Table 4** for each of the variable forcing functions considered. Solutions (4) for u_n are evaluated and plotted in **Fig. 12(a,b)**, **13(a,b), 14(a,b)** and **15(a,b)**.

Fig. 12(a) illustrates u_n for *n* = 1 (namely, solution to Airy's equation) and **Fig. 12(b)** illustrates u_n for *n* = 10, for the three variable forcing functions considered. These figures demonstrate the similarity in qualitative behaviour of the solutions to Airy's and generalized Airy's equations for all forcing functions tested. When *n* =1, solution curves tend to be of parabolic shape with higher curvature than for the case of *n* = 10. This might indicate that for values of *n* higher than 10 the velocity profile tends to be a linearly increasing function.

Fig. 13(a,b), 14(a,b) and **15(a,b)** illustrate u_n for $f(y) = y$, $f(y) = y^2$ and $f(y) = \sin y$, respectively, and the various values of *n*. Again, for visual clarity, the figures group the cases of *n* = 1,2,3 in one graph and *n* = 4,5,10 in another graph. All of these graphs show the relative positions of the solution curves with increasing *n*, and demonstrate the increase in u_n with increasing *n* over the interval $0 \le y \le 1$. This pattern persists for all forcing functions considered. In addition, with increasing *n* the solution curves tend to be closer together, thus indicating that *n* has a greater influence on the solution curves than the form of forcing function.

Table 4. BVP values of c_{1n} and c_{2n} for different values of *n* and variable $f(y)$.

	$f(y) = y$	$f(y) = y^2$	$f(y) = \sin y$
$n = 1$	c_{1n} = 0.2413637726 c_{2n} = 1.486858836	c_{1n} = 0.08664877402 c_{2n} = 1.576183581	c_{1n} = 0.2266598861 c_{2n} = 1.495348129
$n = 2$	c_{1n} = -0.08366104386 c_{2n} = 1.774831441	c_{1n} = -0.2930912902 c_{2n} = 1.879546565	c_{1n} = -0.1037847989 c_{2n} = 1.784893318
$n = 3$	c_{1n} = -0.2797513670 c_{2n} = 2.011925176	c_{1n} = -0.5406919975 c_{2n} = 2.128621374	c_{1n} = -0.3049613350 c_{2n} = 2.023199417
$n = 4$	c_{1n} = -0.4095720003 c_{2n} = 2.218349228	c_{1n} = -0.7200119992 c_{2n} = 2.345085827	c_{1n} = -0.4396598316 c_{2n} = 2.230632534
$n = 5$	c_{1n} = -0.4987323184 c_{2n} = 2.404067516	c_{1n} = -0.8573188742 c_{2n} = 2.539600494	c_{1n} = -0.5335568124 c_{2n} = 2.417229937
$n = 10$	c_{1n} = -0.6486050131 c_{2n} = 3.159834601	c_{1n} = -1.237368590 c_{2n} = 3.329796007	c_{1n} = -0.7060034923 c_{2n} = 3.176404117

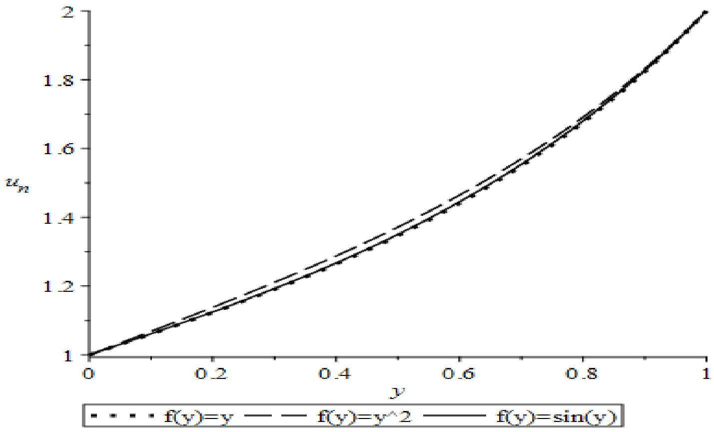

Fig. 12(a) Solutions to Airy's BVP with Variable Forcing Functions $f(y)$

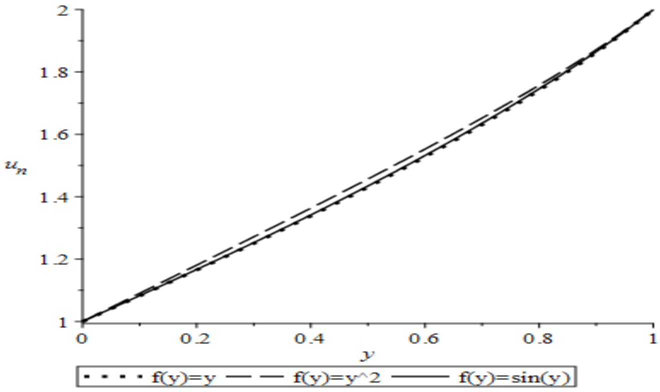

Fig. 12(b) Solutions to Generalized Airy's BVP with Variable Forcing Functions $f(y)$

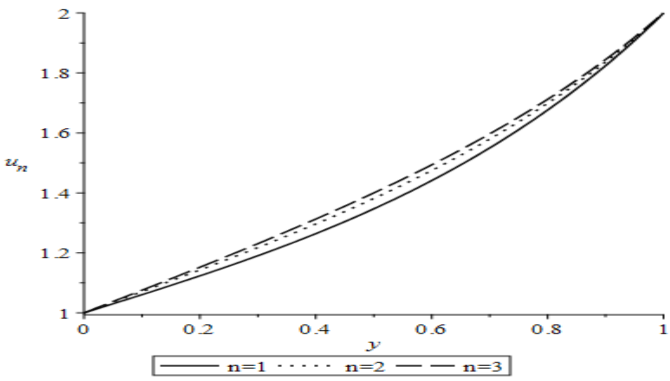

Fig. 13(a) Solutions to Generalized Airy's BVP with $f(y) = y$, $n = 1, 2,$ and 3

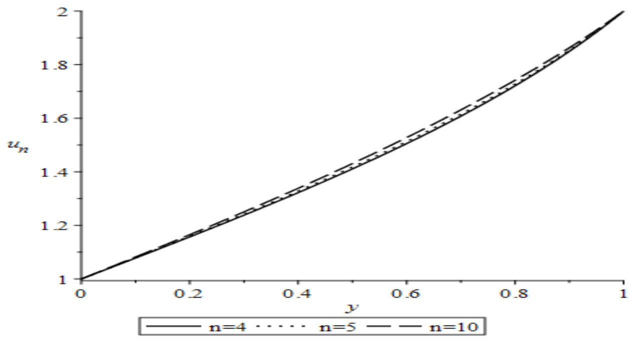

Fig. 13(b) Solutions to Generalized Airy's BVP with $f(y) = y$, $n = 4, 5,$ and 10

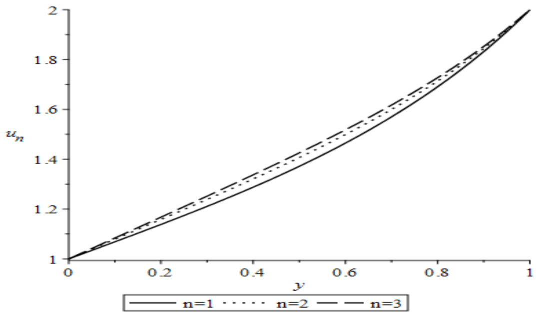

Fig. 14(a) Solutions to Generalized Airy's BVP with $f(y) = y^2$, n = 1, 2, and 3

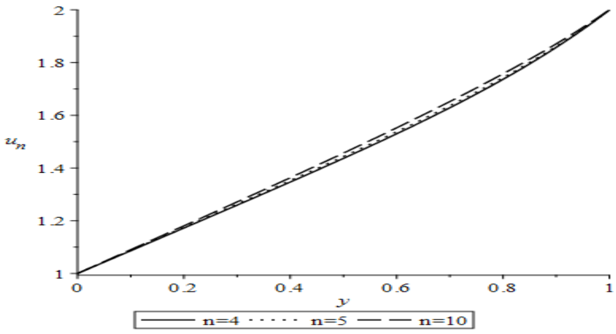

Fig. 14(b) Solutions to Generalized Airy's BVP with $f(y) = y^2$, n = 4, 5, and 10

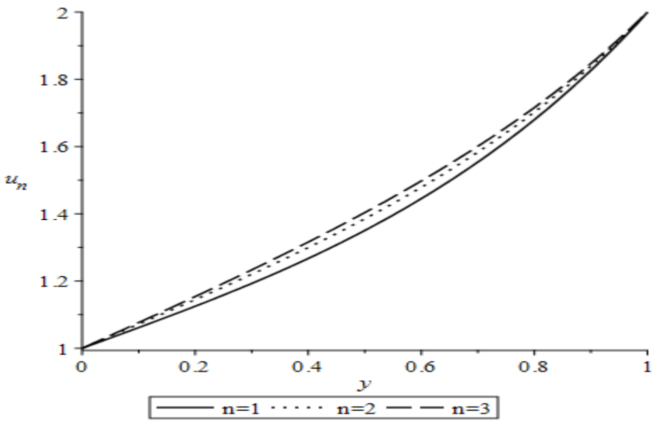

Fig. 15(a) Solutions to Generalized Airy's BVP with $f(y) = \sin y$, n = 1, 2, and 3

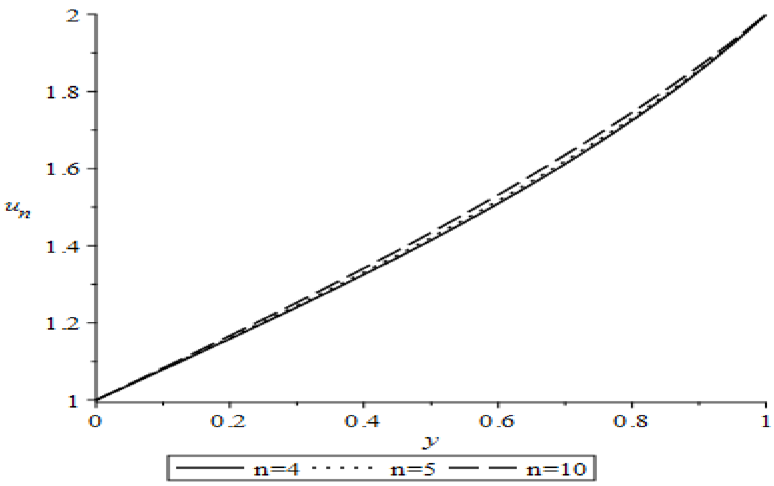

Fig. 15(b) Solutions to Generalized Airy's BVP with $f(y) = \sin y$, n = 4, 5, and 10

VII. CONCLUSION

In this work we provided analysis and computations of the inhomogeneous generalized Airy's ordinary differential equation with Airy's index *n*, subject to initial and boundary conditions. The forcing functions chosen are either constant ($f(y) = \kappa = \mp 1/k$) or variable $(f(y) = y;\ f(y) = y^2;\ f(y) = \sin y)$. The generalized Airy's equation reduces to Airy's equation when n = 1. General solutions to the inhomogeneous generalized Airy's equation have been expressed and evaluated in terms of the generalized Nield-Kuznetsov functions of the first-kind (for constant forcing functions) and second-kinds (for variable forcing functions). When n = 1, the generalized Nield-Kuznetsov functions reduce to the standard Nield-Kuznetsov functions of the first and second kinds.

Solutions have evaluated using computational procedures based on series expressions for the generalized Airy's functions and for the Nield-Kuznetsov functions. Arbitrary constants and the solutions are tabulated or graphed in this work, and support the following conclusions.

(a) Values of the arbitrary constants in the case of initial value problem are independent of the forcing function in Airy's and generalized Airy's equation, but are dependent on Airy's index *n*.

(b) Values of the arbitrary constants in the case of boundary value problem depend on both the forcing function and Airy's index.

(c) For both the initial and boundary value problems with constant or variable forcing functions, solutions are increasing over the interval $0 \le y \le 1$, with exponentially varying solution curves for the initial value problem, regardless of the form of forcing function.

(d) With decreasing *n*, solution curves for the initial value problem experience sharper increase with increasing *y*.

(e) With increasing *n*, solution curves for the boundary value problem experience sharper increase with increasing *y*.

VIII. REFERENCES

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