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Some remarks on convergence of measures

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ABSTRACT
In engineering and applied mathematics, the theory of convergence of probability measures related to stochastic processes plays an important role. In this note, we show that under a suitable condition, the weak convergence of measures is equivalent to setwise convergence of measures.

Keywords **-** weak convergence, setwise convergence

I. INTRODUCTION

A sequence of measures $\{u_n, n = 1, 2, ...\}$ on a sigma algebra B_X of Borel subsets of a metric space X converges weakly to a measure u on B_X if for any bounded continuous function f on X (see [1])

$$\lim_{n\to\infty}\int_X f(x)u_n(dx) = \int_X f(x)u(dx).$$

The weak convergence is equivalent to the following statement: for any $u-$continuity set $A \in B_X$ (i.e. a set such that $u(\partial A) = 0$),

$$\lim_{n\to\infty} u_n(A) = u(A).$$

A natural question arises: when will the setwise convergence holds, that is, for any $A \in B_X$, we always have $\lim_{n\to\infty} u_n(A) = u(A)$? This note gives such a criteria to guarantee the equivalence between the weak convergence and setwise convergence.

II. THE MAIN RESULT

Theorem 1: Let (X, B_X) be any metric space, where B_X is the Borel sigma algebra. Suppose $\{u_n, n = 1, 2, ...\}$ is a sequence of measures on (X, B_X), and u, λ are two finite measures on (X, B_X).Assume that $\{u_n\}$ weakly converge to u and $\sup_n u_n \le \lambda$. Then we have

$$\lim_{n\to\infty} u_n(\Gamma) = u(\Gamma) \text{ for any } \Gamma \in B_X.$$

Proof: We will prove this theorem in two steps. The first step is to show that for any closed set $\Gamma \in B_X$,

$$\lim_{n\to\infty} u_n(\Gamma) = u(\Gamma).$$

To see this, we choose a sequence of bounded and continuous functions $\{f_k(x)\}$ on X such that

$$\lim_{k\to\infty} f_k(x) = 1_\Gamma(x), \text{ for any } x \in X.$$

For example, we can choose

$$f_k(x) = (\exp\{-\rho(x, \Gamma)\})^k,$$

where ρ is the metric defined on X. Now

$$u_n(\Gamma) = \int_X 1_\Gamma(x)u_n(dx) = \int_X \lim_{k\to\infty} f_k(x)u_n(dx)$$
$$= \lim_{k\to\infty}\int_X f_k(x)u_n(dx).$$

What's more,

$$\lim_{n\to\infty} u_n(\Gamma) = \lim_{n\to\infty}\lim_{k\to\infty}\int_X f_k(x)u_n(dx)$$
$$= \lim_{k\to\infty}\lim_{n\to\infty}\int_X f_k(x)u_n(dx) = \lim_{k\to\infty}\int_X f_k(x)u(dx)$$
$$= u(\Gamma)$$

where the second equality comes from the lemma at the end of this section, since $\lim_{k\to\infty}\int_X f_k(x)u_n(dx)$ exists uniformly in n by noticing the following fact

$$\left|\int_X f_k(x)u_n(dx) - \int_X 1_\Gamma(x)u_n(dx)\right|$$
$$\le \int_X |f_k(x) - 1_\Gamma(x)| u_n(dx)$$
$$\le \int_X |f_k(x) - 1_\Gamma(x)| \lambda(dx) \to 0.$$

The second step is to show that for any set $\Gamma \in B_X$,

$$\lim_{n\to\infty} u_n(\Gamma) = u(\Gamma).$$

We will use $\pi - \lambda$ argument to prove this. Let's define

$$\mathrm{A}_1 = \{\Gamma : \Gamma \text{ is closed in } X\},$$
$$\mathrm{A}_2 = \left\{\Gamma : \lim_{n\to\infty} u_n(\Gamma) = u(\Gamma)\right\}.$$

It is obvious that A_1 is a π class. Now we prove that A_2 is a λ class. First, by the definition of weak convergence, we have $\lim_{n\to\infty} u_n(X) = u(X)$, thus the whole space $X \in \mathrm{A}_2$. Second, the set A_2 is obviously closed under the operation of disjoint unions. Third, if $\Gamma_1, \Gamma_2 \in \mathrm{A}_2$ and $\Gamma_1 \supseteq \Gamma_2$, then

$$\lim_{n\to\infty} u_n(\Gamma_1 - \Gamma_2) = \lim_{n\to\infty} [u_n(\Gamma_1) - u_n(\Gamma_2)]$$
$$= u(\Gamma_1) - u(\Gamma_2).$$

Last, for a sequence of non-decreasing sets $\{\Gamma_n\}$,

$$\lim_{n\to\infty} u_n(\bigcup_{m=1}^{\infty} \Gamma_m) = \lim_{n\to\infty}\lim_{k\to\infty} u_n(\bigcup_{m=1}^{k} \Gamma_m)$$
$$= \lim_{n\to\infty}\lim_{k\to\infty} u_n(\Gamma_k) = \lim_{k\to\infty}\lim_{n\to\infty} u_n(\Gamma_k)$$
$$= \lim_{k\to\infty} u(\Gamma_k) = u(\bigcup_{m=1}^{\infty} \Gamma_m),$$

thus $\bigcup_{m=1}^{\infty} \Gamma_m \in \mathrm{A}_2$. Again, the third equality in the above equation is from the lemma below since $\lim_{k\to\infty} u_n(\Gamma_k)$ exists uniformly in n,

$$\left| u_n(\Gamma_k) - u_n\left(\bigcup_{m=1}^{\infty} \Gamma_m\right)\right| = u_n\left(\bigcup_{m=k+1}^{\infty} \Gamma_m\right)$$
$$\le \lambda\left(\bigcup_{m=k+1}^{\infty} \Gamma_m\right) \to 0.$$

Thus A_2 is a λ class containing π class A_1, so $\mathrm{A}_2 \supseteq \sigma(\mathrm{A}_1) = B_X$.
The proof is thus complete.

Lemma 1: Let $f(x,y)$ be a real valued function. Suppose that for every x, the limit $\lim_{y\to y_0} f(x,y)$ exists, and the limit $\lim_{x\to x_0} f(x,y)$ exists uniformly in y. Assume that $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = c$, then

$$\lim_{y\to y_0}\lim_{x\to x_0} f(x,y) = \lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = c.$$

Proof: For any $\varepsilon > 0$, from the fact that $\lim_{x\to x_0} f(x,y) = g(y)$ uniformly in y, we have some $\delta_1(\varepsilon) > 0$ such that for $|x - x_0| < \delta_1(\varepsilon)$,

$$|f(x,y) - g(y)| < \varepsilon.$$

The assumption $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = c$ implies there is $\delta_2(\varepsilon) > 0$ such that for $|x - x_0| < \delta_2(\varepsilon)$,

$$\left|\lim_{y\to y_0} f(x,y) - c\right| < \varepsilon.$$

We now choose a point x_1 such that

$$|x_1 - x_0| < \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}.$$

That $\lim_{y\to y_0} f(x_1,y)$ exists implies there is $\delta_3(\varepsilon) > 0$ such that for $|y - y_0| < \delta_3(\varepsilon)$,

$$\left| f(x_1,y) - \lim_{y\to y_0} f(x_1,y)\right| < \varepsilon.$$

Thus for any $|y - y_0| < \delta_3(\varepsilon)$,

$$|g(y) - c|$$
$$\le \left| g(y) - f(x_1,y) + f(x_1,y) - \lim_{y\to y_0} f(x_1,y) + \lim_{y\to y_0} f(x_1,y) \right|$$
$$< 3\varepsilon.$$

References

[1] P. Billingsley, *Convergence of probability measures* (New York: John Wiley & Sons Inc, 1999).