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Extremal Solutions of Boundary Value Problems Using Fixed Point Theorems

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ABSTRACT
Existence of extremal fixed points of A + B is obtained in ordered Banach spaces. Some applications to two- point boundary value problems in ordinary differential equations are discussed.

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1. INTRODUCTION

Krasnoselskii [2] proved the existence of fixed points of A + B in closed convex Banach spaces while many mathematician had obtained its extremal fixed points in ordered Banach spaces. Here we generalize some results of [1.

Let X be a Banach space and K a cone in X. Let $\leq$ be a partial ordering defined by K i.e. for x, y in X, $x \leq y$ if $y - x \in K$. A cone K is said to be regular, if every increasing and bounded in order sequence has a limit and normal if there exists $N > 0$ such that $0 \leq x \leq y$ implies $\| x \| \leq N \| y \|$. The details about cone and their properties may be found in [1].

Let $x_0, y_0 \in X$ with $x_0 \leq y_0$, the set $[x_0,y_0] = \{x \in X: x_0 \leq x \leq y_0\}$ is called order interval in X.

A mapping T: $D \subset X \to X$ is said to be increasing if $x_1 \leq x_2$ implies $T x_1 \leq T x_2$. T is said to be a nonlinear contraction if there exist a lower semi continuous real function ϕ with $\phi(r) < r$, $r > 0$ satisfying

$\| Tx - Ty \| \leq \phi(\|x - y\|)$, for all x, y in D. (1)

A mapping T is said to be condensing if $\gamma(T(S)) < \gamma(S)$ where $S \subset D$ and γ is Kuratowskii's measure of noncompactness . It is evident that if T is completely continuous then it is condensing.

2. FIXED POINT THEOREMS.

Theorem 2.1: Let $x_0, y_0 \in X$, $x_0 < y_0$ and $A,B:[x_0,y_0] \to X$ satisfy the following conditions:

(H_1) A is a nonlinear contraction,
(H_2) $Ax + By \in [x_0, y_0]$ for $x, y \in [x_0,y_0]$
(H_3) $(I - A)^{-1}B$ is increasing where I denote an identity operator
(H_4) B is semi continuous i.e. $x_n \to x$ strongly $\Rightarrow Bx_n \to Bx$ weakly.

Suppose that the cone K in the Banach space X is regular. Then the mapping A + B has maximal and minimal fixed points in $[x_0, y_0]$.

Proof: Assume $T = (I - A)^{-1}B$, the existence of T is guaranteed by hypothesis (H_1). Claim that T maps $[x_0, y_0]$ into itself. For fixed $y \in [x_0, y_0]$ define a mapping A_y on $[x_0,y_0]$ by

$$A_y(x) = Ax + By \quad (2)$$

where $x \in [x_0, y_0]$. Hypothesis (H_2) implies that A_y maps $[x_0, y_0]$ into itself moreover for $x_1, x_2 \in [x_0,y_0]$

$$\|A_y(x_1) - A_y(x_2)\| \leq \phi(\|x_1 - x_2\|)$$

and hence A_y is a nonlinear contraction. Therefore A_y has a unique fixed point $y' \in [x_0, y_0]$ such that $A_y(y') = y'$. Now for $x \in [x_0, y_0]$,

$Tx = y \Rightarrow Ay + Bx = y \Rightarrow A_x(y) = y$

But A_x has unique fixed point in $[x_0, y_0]$ and hence $Tx \in [x_0, y_0]$. Therefore T maps $[x_0,y_0]$ into itself.

Now consider the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_{n+1} = Tx_n \text{ and } y_{n+1} = Ty_n \quad (3)$$

Hypothesis (H_3) implies that

$$x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y_n \leq \ldots \leq y_1 \leq y_0 \quad (4)$$

Since K is regular and sequences $\{x_n\}$ and $\{y_n\}$ are bounded in order, these sequences converge to x^* and y^* respectively. Hypothesis (H_4) implies that T is semi continuous and so $Tx^* = x^*$ and $Ty^* = y^*$. The fixed points of T are also fixed points of A + B. Therefore x^* and y^* are fixed points of A+B.

If x is any fixed point of A + B then $x_0 \leq x \leq y_0$. Since T is increasing, $x_1 \leq x \leq y_1$, and by induction $x_n \leq x \leq y_n$ for n = 0,1,2,3…Taking the limits, we obtain $x^* \leq x \leq y^*$. Thus x* and y* are minimal and maximal fixed points of A + B. This completes the proof.

Corollary 2.1: Let the conditions of Theorem 2.1 be satisfied. Suppose A + B has only one fixed point x in $[x_0,y_0]$. Then for any $u_0 \in [x_0,y_0]$, the sequence

$$u_{n+1} = Au_{n+1} + Bu_n. \qquad (5)$$

converges to x. i.e. $\| u_n - x \| \to 0 \quad (n \to \infty)$.

Proof: Define a mapping T on $[x_0, y_0]$ as in Theorem 2.1. Then the sequence (5) can be obtained as the successive iterates

$$u_{n+1} = Tu_n.$$

Since $x_0 \leq u_0 \leq y_0$ and T is increasing, $x_n \leq u_n \leq y_n$. By hypothesis, x is the only fixed point of T and hence x* = x = y* where x* and y* are limits of sequences $\{Tx_n\}$ and $\{Ty_n\}$. Cone K is regular implies that K is normal. So by normality of cone and Theorem 2.1, it follows that $u_n \to x$.

Theorem 2.2: Let the conditions (H_1) and (H_2) of Theorem 2.1 be satisfied. If B is completely continuous and K is normal then A + B has a fixed point in $[x_0, y_0]$.

Proof: Define a mapping T as in Theorem 2.1. Therefore T maps $[x_0, y_0]$ into itself. Since $(I - A)^{-1}$ is continuous and B is completely continuous, T is also completely continuous. Moreover K is normal and hence order interval $[x_0, y_0]$ is closed convex and bounded. Existence of fixed point of T is now guaranteed by Schauder's theorem. Hence A + B has a fixed point.

Theorem 2.3: Assume conditions (H_1) – (H_3) of theorem 2.1. If B is condensing and K is normal then A + B has a fixed point.

Proof: Define a mapping T as in Theorem 2.1. For fixed $u_0 \in [x_0, y_0]$ define a sequence $\{u_n\}$ by $u_{n+1} = Tu_n$. Let $S = \{u_0, u_1, u_2 \ldots\}$

$$\therefore \quad S = T(S) \cup \{u_0\}.$$

Since $(I - A)^{-1}$ is continuous and B is condensing, T is also condensing.

Hence $\gamma(S) = \gamma(T(S)) < \gamma(S)$. Therefore $\gamma(S) = 0$. This implies that S is relatively compact. Hence there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to x^*$. But K is normal and so $u_n \to x^*$. Taking limits $n \to \infty$ in $u_n = Tu_{n-1}$, we get $x^* = Tx^*$ since T is continuous. Hence x* is a desired fixed point of A + B.

Theorem 2.4: Assume conditions (H_1) - (H_4) of Theorem 2.1. If B is condensing and K is normal then A + B has minimal and maximal fixed points.

Proof: Define a mapping T as in the proof of Theorem 2.1. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3) converge respectively to x* and y*. It is obvious that $x^* \leq y^*$ and x* ,y* are fixed points of T. By similar procedure as expressed in Theorem 2.1, it can be proved that x* and y* are minimal and maximal fixed points of A+B.

Corollary 2.2: Let the condition of Theorem 2.4 be satisfied. Suppose A + B has only one fixed point $x \in [x_0, y_0]$. Then for any $u_0 \in [x_0, y_0]$ the sequence of iterates defined by (5) converges to x. i.e. $\|u_n - x\| \to 0$ as $(n \to \infty)$.

The proof of this corollary is similar to that of corollary 2.1.

Remark-1: Consider the condition

(H_5) : A is linear and A^k, for some $k \in N$, is nonlinear contraction on $[x_0, y_0]$.

If A_y is defined by (2) then for any $x \in [x_0, y_0]$ using linearity of A, it follows that

$$A_y^k(x) = A^k x + (I + A + A^2 + \ldots + A^{k-1}) By.$$

Thus for $x_1, x_2 \in [x_0, y_0]$,

$$\| A_y^k(x_1) - A_y^k(x_2) \| \leq \phi(\|x_1 - x_2\|)$$

which shows that A_y^k is a nonlinear contraction and hence A_y has a unique fixed point. This guarantees the existence of mapping T as defined in the proof of Theorem 2.1. The definition of A_y and T shows that T maps $[x_0, y_0]$ into itself. Thus Theorem 2.1 holds even if the condition (H_1) is replaced by (H_5).

Remark 2: Theorem 2.1.1 of [2] appears as a special case of Theorem 2.1, which may be obtained by putting $A \equiv \theta$.

Remark 3: The condition (H_3) had been imposed in the Theorem 3 of [3]. This condition can be excluded to obtain the result of Theorem 3 of [3] as seen from Theorem 2.2. Our approach to prove the Theorem is quite simple and different from that of [3].

Remark 4: Corollary 2.1.1 of [2] is a special case of Corollaries 2.1 and 2.2.

Remark 5: Theorem 2.2 holds even if the condition (H1) is replaced by (H5).

3. APPLICATIONS

Let X = C[I , R] be the set of continuous real valued function defined on I = [0, 1] with supremum norm and $K = \{x \in X : x(t) \geq 0, 0 \leq t \leq 1\}$. Then X is a Banach space and K is a cone in X. Moreover K is normal and regular. Consider the two-point boundary value problem of ordinary differential equation

$$-x'' = \lambda f(t, x) + \mu g(t, x) \quad (6)$$

$$x(0) = 0 = x(1) \quad (7)$$

where λ, μ are parameters and f, g: I x X $\rightarrow$ X. The functions u and v, in $C^{(2)}$ [I, R] are said to be respectively lower and upper solutions of (6) if

$$-u''(t) \geq \lambda f(t, u(t)) + \mu g(t, u(t)) \quad (8)$$

and

$$-v''(t) \leq \lambda f(t, u(t)) + \mu g(t, u(t)) . \quad (9)$$

We need the following assumptions:

(A_1) f (t, x) satisfies the Lipschitz condition in x with Lipschitz constant L i.e. there is a constant L > 0 such that

$$|f(t, x_1) - f(t, x_2)| \leq L |x_1 - x_2|.$$

(A_2) g(t, x) is continuous on $0 \leq t \leq 1$.

(A_3) f(t, x) and g(t, x) are increasing with respect to x i.e. for $0 \leq t \leq 1, 0 \leq x_1 \leq x_2$,

$$f(t, x_1) \leq f(t, x_2)$$

and $$g(t, x_1) \leq g(t, x_2) .$$

It is obvious that $x_{\lambda,\mu}(t) \equiv 0$ is a trivial solution of problem (6) – (7) for any values of λ and μ.

Theorem 3.1: Assume (A_1) – (A_3). Suppose that the function u and v are respectively the lower and upper solutions of equation (6). Further if $\lambda L < 8$, then the equations (6) – (7) have minimal and maximal solutions in [u, v].

Proof: It is well known that the solution of problem (6) – (7) is equivalent to the solution of integral equation

$$x(t) = \int_0^1 G(t,s)[\lambda f(s, x(s)) + \mu g(s, x(s))] ds \quad (10)$$

where G(t, s) is the Green's function of differential operator $-x''$ with respect to boundary conditions x(0) = 0 = x(1) given by

$$G(t, s) = \begin{cases} t(1-s) & \text{for } 0 \leq t \leq s \leq 1 \\ s(1-t) & \text{for } 0 \leq s \leq t \leq 1 \end{cases} \quad (11)$$

It is easy to show that

$$\int_0^1 G(t,s)\, ds = \frac{t(1-t)}{2} \leq \frac{1}{8}.$$

Define

$$Ax(t) = \lambda \int_0^1 G(t,s)\, f(s, x(s)) ds$$

and $$Bx(t) = \mu \int_0^1 G(t,s)\, g(s, x(s))\, ds$$

For any x, y in [u, v];

$$\|Ax - Ay\| = \sup_{t\in I} |Ax(t) - Ay(t)|,$$
$$\leq \sup_{t\in I} \lambda \int_0^1 G(t,s) | f(s, x(s)) - f(s, y(s)) | ds$$
$$\leq \sup_{t\in I} \lambda L \int_o^1 G(t,s) | x(s) - y(s)| ds$$
$$\leq \frac{\lambda L}{8} \|x - y\|$$

Since $\lambda L < 8$, A becomes a contraction mapping on [u, v] and so $(I - A)^{-1}$ exist. Hypothesis (A_3) implies that $(I - A)^{-1}B$ is increasing, (A_2) implies that B is completely continuous and hence B is condensing. Applying the theory of differential inequality to (8) and (9) we see that

$$u(t) \leq \lambda \int_0^1 G(t,s) f(s, u(s)) ds + \mu \int_0^1 G(t,s) g(s, u(s)) ds$$
$$\leq \lambda \int_0^1 G(t,s) f(s, x(s)) ds + \mu \int_0^1 G(t,s) g(s, y(s)) ds$$
$$= Ax(t) + By(t)$$
$$\leq \lambda \int_0^1 G(t,s) f(s, v(s) ds + \mu \int_0^1 G(t,s) g(s, v(s)) ds$$
$$\leq v(t)$$

where x, y are in [u, v]. Therefore A + B maps [u, v] into itself. Theorem 2.4 asserts that A + B has minimal and maximal fixed points in [u, v] which are desired solutions of equations (6) – (7).

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