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On contact slant submanifolds in Kenmotsu space forms

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Abstract : **In this paper a sharp estimate of the maximum Ricci curvature function for a special contact slant submanifold M in a Kenmotsu space form $\overline{M}(c)$, in terms of the main extrinsic invariant namely the squared mean curvature has been obtained. If submanifold M satisfies the equality case identically, then it is minimal.**

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I. INTRODUCTION

The study of slant submanifold of an almost Hermitian manifold was started by B. Y. Chen ([1], [2]). Further Chen and Tazawa ([3], [4]), furnished examples of slant submanifolds of C^2 and C^4, while Maeda, Ohnita and Udagawa [5], studied slant submanifolds of Kaehler manifold. The notion of slant submanifold of almost contact metric manifold was introduced by A. Lotta [6]. A class of manifolds was studied and characterized by K. Kenmotsu [7], and the structure so obtained is nowadays known as Kenmotsu structure. In general a Kenmotsu structure is not Sasakian. A Kenmotsu space form is a Kenmotsu manifold with constant ϕ-holomorphic sectional curvature c, denoted by $\overline{M}(c)$. Recently, K. Arslan et. al . [8], have established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in Kenmotsu space forms. A (2m+1)-dimensional Riemannian manifold $\overline{M}$ is said to be an almost contact metric manifold if there exist structure tensors $(\phi.\xi,\eta,g)$, where ϕ is a (1, 1) tensor field, ξ a vector field, η a 1-form and g the Riemannian metric on $\overline{M}$ satisfying [9]:

(1.1) $\phi^2 X = -X + \eta(X)\xi$

(1.2) $\phi\xi = 0$

(1.3) $\eta(\xi) = 1$

(1.4) $\eta(\phi X) = 0$

(1.5) $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$

(1.6) $\eta(X) = g(X,\xi)$

for any $X, Y \in \overline{TM}$, where $\overline{TM}$ denotes the Lie algebra of vector fields on $\overline{M}$. An almost contact metric manifold $\overline{M}$ is called a Kenmotsu manifold if [7],

(1.7) $(\overline{\nabla}_X \phi)Y = -g(X,\phi Y)\xi - \eta(Y)\phi X$ and $\overline{\nabla}_X \xi = X - \eta(X)\xi$

Where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

The curvature tensor $\overline{R}$ of a Kenmotsu space form $\overline{M}(c)$ is given by [7],

$$\overline{R}(X,Y)Z = \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi - g(\phi X,Z)\phi Y + g(\phi Y,Z)\phi X + 2g(X,\phi Y)\phi Z\} \tag{1.8}$$

for all $X,Y,Z \in \overline{TM}$.

Let M be an n-dimensional submanifold of a Kenmotsu space form $\overline{M}(c)$ of constant ϕ-sectional curvature c. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$, and ∇ the Riemannian connection on M, respectively. We denote by h the second fundamental form of

M and by R and $\overline{R}$ the curvature tensors of the submanifold M and Kenmotsu space form $\overline{M}(c)$, respectively. Then the equation of Gauss is given by

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \tag{1.9}$$

for all $X,Y,Z,W \in TM$.

Let $p \in M$ and $\{e_1,e_2,...,e_{2m+1}\}$ be an orthonormal basis of the tangent space $\overline{T_pM}$, such that $e_1,e_2,\ldots,e_n$ are tangent to M at p. Let H denote the mean curvature vector, that is $H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)$. Now we set

$$h_{i,j}^{r} = g(h(e_i,e_j),e_r), \quad i,j = \{1,2,...,n\}, \quad r \in \{(n+1),..,(2m+1)\} \tag{1.10}$$

For any vector field X tangent to M, we put $\phi X = PX + FX$, where PX and FX denote the tangential and normal components of ϕX respectively.

II. CONTACT SLANT SUBMANIFOLDS

K. Arslan et. al. [8] have established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in Kenmotsu space forms. We recall the following theorem for our ready reference.

Theorem [8]: Let $\overline{M}(c)$ be a (2m+1)-dimensional Kenmotsu space form and M an n-dimensional submanifold tangent to ξ. Then

(i) for each unit vector $X \in T_pM$ orthogonal to ξ,

$$Ric(X) \le \frac{1}{4}\left\{(n-1)(c-3) + \frac{1}{2}\left(3\|PX\|^2 - 2\right)(c+1) + n^2\|H\|^2\right\} \tag{2.1}$$

(ii) A unit tangent vector X orthogonal to ξ at p satisfies the equality case of (2.1) if and only if

$$(2.2)\qquad \begin{cases} h(X,Y)=0,\ \forall Y \in T_pM \ \textit{orthogonal to}\ X \\ 2h(X,X)=nH(p) \end{cases}$$

A submanifold M tangent to ξ in a Kenmotsu space form $\overline{M}(c)$ is said to be contact slant submanifold if for any $p \in M$ and any $X \in T_pM$ linearly independent on ξ, the angle between ϕX and T_pM is a constant θ, called the slant angle of M. A contact slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

A particular case of contact slant submanifolds, which we call special contact slant submanifolds ([10], [13]). A proper contact θ-slant submanifold is said to be special contact slant if

$$(\nabla_X P)Y = g(Y,PX)\xi - \eta(Y)PX$$

Any 3-dimensional proper contact slant submanifold of a Kenmotsu manifold is a special contact slant submanifold [10].

We denote by $\mathbf{R}$ the maximum Ricci curvature function on M [11], defined by

$$\mathbf{R}(p) = \max\{Ric(u) \mid u \in T_p^1M\},\ p \in M$$

where $T_p^1M = \{u \in T_pM \mid g(u,u)=1\}$.

If $n=3$, $\mathbf{R}$ is the Chen's first invariant δ_M introduced in [12]. For $n>3$, $\mathbf{R}$ is the Chen's invariant $\delta(n-1)$.

In this section, we establish an inequality for the Chen invariant $\mathbf{R}$ and show that any special contact slant submanifold which satisfies the equality case identically is minimal.

Now we show

Theorem (2.1): Let M be an (n+1)-dimensional special contact slant submanifold in a (2n+1)-dimensional Kenmotsu space form $\overline{M}(c)$ of constant ϕ-sectional curvature c. Then for each unit vector X orthogonal to ξ, we have

$$(2.3)\qquad \mathbf{R} \le \frac{1}{4}\{(n+1)^2\|\mathrm{H}\|^2 + \mathrm{n}(\mathrm{c}-3) + \frac{1}{2}(3\cos^2\theta - 2)(\mathrm{c}+1)\}$$

If M satisfies the equality case in (2.3) identically, then M is a minimal submanifold.

Proof: The inequality (2.3) is an immediate consequence of the inequality (2.1). To ascertain minimality, we assume that M is a special contact slant submanifold of Kenmotsu space form $\overline{M}(c)$, which satisfies the equality case of (2.3) at a point $p \in M$. Now without loss of generality we may choose an orthonormal basis $\{e_0 = \xi, e_1, \ldots, e_n\}$ of T_pM such that $\mathbf{R}(p) = Ric(e_n)$. Then by (2.2) it follows that

$$(2.4)\qquad h(e_i,e_n)=0,\ \forall\ i \in \{0,\ldots,n-1\};\quad h(e_n,e_n) = \sum_{i=1}^{n-1} \mathrm{h}(\mathrm{e_i},\mathrm{e_i})$$

where $\{e_0 = \xi, e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ is the orthonormal basis with $e_{n+j} = (\sec\theta)Pe_j$, $j \in \{1,2,\ldots,n\}$ and e_{n+1} parallel to the mean curvature vector $H(p)$.

Let A denote the Weingarten map of M in $\overline{M}(c)$. Then it is easy to show that $A_{\phi X}Y = A_{\phi Y}X$ for all vector fields X, Y tangent to M (see [10]). Hence we have $h_{i,j}^{n+k} = h_{i,j}^{n+j} = h_{i,j}^{n+i}$ for any $i, j, k \in \{1, 2, .. , n\}$. Now using equations (2.4) a simple computation yields $H(p) = 0$, therefore M is a minimal submanifold.

We have the following characterization of n-dimensional special slant submanifolds in the sphere S^{2n+1} which satisfy the equality case of (2.3) identically.

Theorem (2.2): Let M be an (n+1)-dimensional special contact slant submanifold of the (2n+1)-dimensional sphere S^{2n+1}. Then M satisfies the equality case of (2.3) identically if and only if M is minimal and ruled.

Proof: The theorem (2.1) implies that M is a minimal submanifold. From the theorem [8], it is obvious that the equality case of (2.3) holds identically if and only if the relative null space

$$\mathbf{N}_p = \{X \in T_pM \mid h(X,Y) = 0, \forall\ Y \in T_pM\} \neq \{0\}$$

at any $p \in M$.

Let U_l denotes interior of the subset consisting of points in M such that the relative null space at p has dimension l. It follows that for some l, $U_l \neq \varphi$, where $l \in \{1, ..., n\}$.

Let $X, Y \in \mathcal{N}$, and Z any vector field tangent to M. By applying Codazzi equation

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Z h)(X,Y),$$

we get $h(\nabla_X Y, Z) = 0$. Then $\mathcal{N}$ is integrable on U_l and each leaf of $\mathcal{N} \mid U_l$ is an l-dimensional totally geodesic submanifold of S^{2n+1}. Thus submanifold M contains a geodesic of S^{2n+1} through each point $p \in U_l$.

Since M is the union of the closure of all U_l, we conclude by continuity that M contains a geodesic of the ambient space through each point in M. Hence, M is minimal and ruled.
The converse follows from straightforward computation.

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