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Free Flexural Vibration Analysis of Simply Supported Kirchhoff Plates by Ritz and Galerkin Methods: a Comparative Approach

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Abstract: In this study, the free vibration analysis of simply supported rectangular Kirchhoff plates of various aspect ratios was carried out using the Ritz and Galerkin methods in the purpose of comparing the two methods. The same systematically constructed comparison functions were used in both methods, and the procedures were implemented with the help of purposefully developed computer programmes. The numbers of terms in the shape functions used were 1, 4, 9, 16, 25 and 36, and the first six frequency parameters were retained due to the deteriorating accuracy from the lower to the upper ends of the eigenvalue spectrum. Both methods gave identical results which also were in excellent agreement with the exact solutions, irrespective of the aspect ratio considered. However, the most consistent results were obtained in the case of a square plate. From the CPU time efficiency study carried out, it was observed that the Galerkin method is more time-efficient than the Ritz method for lower numbers of terms in the shape functions, while the Ritz method exhibited much more time efficiency than its counterpart for higher numbers of terms in the shape functions.
Keywords: Frequency parameter, free vibration, Kirchhoff plate, simply supported plate, polynomial trial function.

I. INTRODUCTION

Kirchhoff plates are thin plates under small deformations fulfilling a number of hypotheses commonly known as Kirchhoff hypotheses [1, 2, 3, 4]:

(i) The constituent plate's material is elastic, homogenous and isotropic.
(ii) The plate is initially flat.
(iii) The deflection of the middle plane of the plate is small compared to its thickness.
(iv) Points of the plate lying initially on a normal to the middle plane of the plate remain on the normal to the middle surface of the plate after bending.
(v) The normal stresses in the transverse direction are negligible comparatively to the other stress components.
(vi) The middle surface of the plate remains unstrained after bending.

Such plates are extensively used in the various branches of engineering as a result of their load-carrying capacity and the economy in materials [5]. Their applications extend to numerous architectural structures, bridges, hydraulic structures, machine parts etc., and they are often subjected to dynamic loads. The vibration analysis of Kirchhoff plates of various shapes and boundary conditions is therefore of paramount importance in order to avoid their failure. The present study focusses on the free vibration analysis of simply supported rectangular Kirchhoff plates.

Abundant literature on the dynamic analysis of thin rectangular plates exists. However, exact analytical solutions for the plate problems are available for few boundary conditions which include all combinations in which two opposite edges are simply supported, and those with one edge free to slide while the rotation is restrained and, on the opposite side, simply supported or sliding [6]. Because of the tediousness of the exact solutions (in cases where they exist) and their absence for numerous edge conditions, several approximate methods of plates' vibration analysis have been devised by researchers. The Ritz and Galerkin methods which have the merit to provide close form solutions are examples of such approximate approaches. Both methods are applicable to the dynamic analysis of rectangular plates of various boundary conditions, provided that appropriate shape functions are obtained. In this work, the two methods are used to carry out the free vibration analysis of simple supported Kirchhoff plates of various aspect ratios using similar shape functions in the purpose of comparing them in terms of accuracy and efficiency.

In general the two methods are not mathematically equivalent: the Ritz method is a versatile method of global as well as piece-wise approximations to the solution of variational problems in solid mechanics and requires the knowledge of the energy functional; the Galerkin method is also known for obtaining approximate solutions but of (partial) differential equations and thus requires the knowledge of the (partial) differential equation of the system [7]. It is well documented that the Ritz method is applicable to only variational formulations without invoking any (partial) differential equations of the system to analyse. That is why it is referred to as a direct variational method. The Galerkin method which is weighted residual method invoking (partial) differential equations of the systems to analyse, is generally referred to as an indirect method. The Galerkin method may be more preferable than its counterpart, the Ritz method, if it is more convenient to work with the governing differential equations rather than with the energy functional. Moreover, there are problems for which no satisfactory variational principle has been formulated, but for which a set of governing differential equations exists. This suggests that the Galerkin method has broader application than the Ritz approach [3].

Theoretically, in order to obtain a correct solution, the Ritz method requires a set of admissible functions which must not violate the essential or geometric or kinetic boundary conditions and should also be linearly independent and complete [8, 9, 10, 11, 12, 13]. If comparison functions (i.e. the ones satisfying both kinetic and natural boundary conditions) are used in Ritz method, more accurate eigenvalues are expected. The Galerkin method also requires linearly independent basis functions but of the class of comparison functions. The accuracy and convergence of the solutions generated by the two methods strongly depend on the choice of the trial functions which is based on an intuitive idea of what the true deformation of the structure looks like, and this is actually one of the weaknesses of both methods.

The present work intends to construct polynomial comparison functions in a systematic way, bearing in mind their number in the series and their degree. The level of round-off errors inherent to polynomial trial functions shall probably be eased by the use of symbolic computing offered by Mathematica software.

II. METHODOLOGY

2.1 Construction of the Polynomial Trial Functions

Consider a simply supported rectangular Kirchhoff plate of side dimensions a and b and thickness h. Trial functions of the class of comparison functions are to be constructed which will be used in the analysis of the plates using both the Ritz and Galerkin methods. To this end, the shape is considered in the form:

$$W(x,y) = \sum_{i}^{m}\sum_{k}^{n} C_{ik} F_i(x) G_k(y) \tag{1}$$

$F_i(x)$ is a polynomial function of purely x, x being the coordinate in the direction of the dimension a of the plate. $G_k(y)$ is a polynomial function of purely y, y being the coordinate in the direction of the dimension b of the plate. C_{ik} are unknown coefficients.

Let the polynomial trial function in x-direction be sought in the form: [14]

$$F_i = a_1^{(i)} x^{i-1} + a_2^{(i)} x^i + a_3^{(i)} x^{i+1} + a_4^{(i)} x^{i+2} \tag{2}$$

where i is a non-nil integer number; the coefficients $a_k^{(i)}$ (k = 1, 2, 3, 4) are obtained by imposing the boundary conditions (geometrical and statical) to the trial functions.

For simply supported ends, the boundary conditions are given as:
$F_i(0) = F_i''(0) = F_i(a) = F_i''(a) = 0$

Imposing the boundary conditions on $F_i(x)$, i = 1, 2, 3, 4, 5, 6, 7, and making use of the dimensionless parameters $\xi = x/a$ and $\eta = y/b$ yield the first six non-null comparison functions in x-direction below:

$$F_1(\xi) = \xi - 2\xi^3 + \xi^4 \tag{3}$$

$$F_2(\xi) = \xi^3 - \frac{7}{4}\xi^4 + \frac{3}{4}\xi^5 \tag{4}$$

$$F_3(\xi) = \xi^3 + \xi^4 - \frac{21}{5}\xi^5 + \frac{11}{5}\xi^6 \tag{5}$$

$$F_4(\xi) = \xi^4 + \xi^5 - \frac{13}{3}\xi^6 + \frac{7}{3}\xi^7 \tag{6}$$

$$F_5(\xi) = \xi^5 + \xi^6 - \frac{31}{7}\xi^7 + \frac{17}{7}\xi^8 \tag{7}$$

$$F_6(\xi) = \xi^6 + \xi^7 - \frac{9}{2}\xi^8 + \frac{5}{2}\xi^9 \tag{8}$$

In a similar manner the following six consecutive comparison functions are obtained in y-direction:

$$G_1(\eta) = \eta - 2\eta^3 + \eta^4 \tag{9}$$

$$G_2(\eta) = \eta^3 - \frac{7}{4}\eta^4 + \frac{3}{4}\eta^5 \tag{10}$$

$$G_3(\eta) = \eta^3 + \eta^4 - \frac{21}{5}\eta^5 + \frac{11}{5}\eta^6 \tag{11}$$

$$G_4(\eta) = \eta^4 + \eta^5 - \frac{13}{3}\eta^6 + \frac{7}{3}\eta^7 \tag{12}$$

$$G_5(\eta) = \eta^5 + \eta^6 - \frac{31}{7}\eta^7 + \frac{17}{7}\eta^8 \tag{13}$$

$$G_6(\xi) = \eta^6 + \eta^7 - \frac{9}{2}\eta^8 + \frac{5}{2}\eta^9 \tag{14}$$

2.2 Ritz Direct Variational Method

In terms of the non-dimensional coordinates, (1) can be rewritten as:

$$W(\xi,\eta) = \sum_{i}^{m}\sum_{k}^{n} C_{ik}F_i(\xi)G_k(\eta) \tag{15}$$

For convenience, (15) can be put in the form: [14]

$$W(\xi,\eta) = \sum_{j}^{p} C_j w_j(\xi,\eta) \tag{16}$$

where p = m x n;

$C_1 = C_{11}, C_2 = C_{12}, C_3 = C_{13}, \ldots, C_n = C_{1n}, C_{n+1} = C_{21}, \ldots, C_{2xn} = C_{2n}, C_{2xn+1} = C_{31}, \ldots, C_p = C_{mn}$;

$w_1(\xi,\eta) = F_1(\xi)G_1(\eta), w_2(\xi,\eta) = F_1(\xi)G_2(\eta), w_3(\xi,\eta) = F_1(\xi)G_3(\eta), \ldots, w_n(\xi,\eta) = F_1(\xi)G_n(\eta), w_{n+1}(\xi,\eta) = F_2(\xi)G_1(\eta), \ldots, w_{2xn}(\xi,\eta) = F_2(\xi)G_n(\eta), w_{2xn+1}(\xi,\eta) = F_3(\xi)G_1(\eta), \ldots, w_p(\xi,\eta) = F_m(\xi)G_n(\eta)$.

In matrix form, (16) can be put as:

$$W(\xi,\eta) = M\,C^T \tag{17}$$

where:

$M = \begin{bmatrix} w_1 & w_2 & \ldots & w_p \end{bmatrix}$ and $C = \begin{bmatrix} c_1 & c_2 & \ldots & c_p \end{bmatrix}$; the superscript T refers to matrix transpose.

The plate's maximum total potential energy will be given by:

$$U_{max} = \frac{1}{2}\frac{Db}{a^3}\int_0^1\int_0^1 \left[W_{\xi\xi}^2 + \alpha^4 W_{\eta\eta}^2 + 2\mu\alpha^2 W_{\xi\xi}W_{\eta\eta} + 2(1-\mu)\alpha^2 W_{\xi\eta}^2\right]d\xi d\eta - \frac{1}{2}\omega^2\rho hab\int_0^1\int_0^1 W^2(\xi,\eta)d\xi d\eta \tag{18}$$

where α is the side ratio a/b and the subscripts ξ and η refer to differentiation with respect to the subscript and the number of times the subscript appears denotes the order of differentiation.

Substitution of (17) into (18) gives:

$$\Pi = \frac{1}{2}\frac{Db}{a^3}\int_0^1\int_0^1 [\left(M_{\xi\xi}C^T\right)^T M_{\xi\xi}C^T + \alpha^4\left(M_{\eta\eta}C^T\right)^T M_{\eta\eta}C^T + 2\mu\alpha^2\left(M_{\eta\eta}C^T\right)^T M_{\xi\xi}C^T + 2(1-\mu)\alpha^2\left(M_{\xi\eta}C^T\right)^T M_{\xi\eta}C^T]d\xi d\eta - \frac{1}{2}\omega^2\rho hab\int_0^1\int_0^1 (MC^T)^T MC^T d\xi d\eta \tag{19}$$

(19) can further be put as:

$$\Pi = \frac{1}{2}\frac{Db}{a^3}C[A_1 + \alpha^4 A_2 + 2\mu\alpha^2 A_3 + 2(1-\mu)\alpha^2 A_4 - \lambda^2 B]C^T \tag{20}$$

where:

$A_1 = \int_0^1\int_0^1 M_{\xi\xi}^T M_{\xi\xi}d\xi d\eta$; $A_2 = \int_0^1\int_0^1 M_{\eta\eta}^T M_{\eta\eta}d\xi d\eta$; $A_3 = \int_0^1\int_0^1 M_{\eta\eta}^T M_{\xi\xi}d\xi d\eta$;

$A_4 = \int_0^1\int_0^1 M_{\xi\eta}^T M_{\xi\eta}d\xi d\eta$; $B = \int_0^1\int_0^1 M^T M d\xi d\eta$ and $\lambda^2 = \frac{\rho h\omega^2 a^4}{D}$

The minimisation of the energy functional given in (20) as required by the Ritz method yields: $\frac{\partial\Pi}{\partial C} = 0$

It follows that:

$$HC^T = 0 \tag{21}$$

where: $H = [A_1 + \alpha^4 A_2 + 2\mu\alpha^2 A_3 + 2(1-\mu)\alpha^2 A_4 - \lambda^2 B]$

A_1, A_2, A_3, A_4 and B are evaluated as follows:

$$A_1 = \int_0^1 \int_0^1 M_{\xi\xi}^T M_{\xi\xi} d\xi d\eta = \int_0^1 \int_0^1 \begin{bmatrix} w_{1\xi\xi}w_{1\xi\xi} & w_{1\xi\xi}w_{2\xi\xi} & \cdots & w_{1\xi\xi}w_{p\xi\xi} \\ w_{2\xi\xi}w_{1\xi\xi} & w_{2\xi\xi}w_{2\xi\xi} & \cdots & w_{2\xi\xi}w_{p\xi\xi} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ w_{p\xi\xi}w_{1\xi\xi} & w_{p\xi\xi}w_{2\xi\xi} & \cdots & w_{p\xi\xi}w_{p\xi\xi} \end{bmatrix} d\xi d\eta \tag{22}$$

$$A_2 = \int_0^1 \int_0^1 M_{\eta\eta}^T M_{\eta\eta} d\xi d\eta = \int_0^1 \int_0^1 \begin{bmatrix} w_{1\eta\eta}w_{1\eta\eta} & w_{1\eta\eta}w_{2\eta\eta} & \cdots & w_{1\eta\eta}w_{p\eta\eta} \\ w_{2\eta\eta}w_{1\eta\eta} & w_{2\eta\eta}w_{2\eta\eta} & \cdots & w_{2\eta\eta}w_{p\eta\eta} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ w_{p\eta\eta}w_{1\eta\eta} & w_{p\eta\eta}w_{2\eta\eta} & \cdots & w_{p\eta\eta}w_{p\eta\eta} \end{bmatrix} d\xi d\eta \tag{23}$$

$$A_3 = \int_0^1 \int_0^1 M_{\eta\eta}^T M_{\xi\xi} d\xi d\eta = \int_0^1 \int_0^1 \begin{bmatrix} w_{1\eta\eta}w_{1\xi\xi} & w_{1\eta\eta}w_{2\xi\xi} & \cdots & w_{1\eta\eta}w_{p\xi\xi} \\ w_{2\eta\eta}w_{1\xi\xi} & w_{2\eta\eta}w_{2\xi\xi} & \cdots & w_{2\eta\eta}w_{p\xi\xi} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ w_{p\eta\eta}w_{1\xi\xi} & w_{p\eta\eta}w_{2\xi\xi} & \cdots & w_{p\eta\eta}w_{p\xi\xi} \end{bmatrix} d\xi d\eta \tag{24}$$

$$A_4 = \int_0^1 \int_0^1 M_{\xi\eta}^T M_{\xi\eta} d\xi d\eta = \int_0^1 \int_0^1 \begin{bmatrix} w_{1\xi\eta}w_{1\xi\eta} & w_{1\xi\eta}w_{2\xi\eta} & \cdots & w_{1\xi\eta}w_{p\xi\eta} \\ w_{2\xi\eta}w_{1\xi\eta} & w_{2\xi\eta}w_{2\xi\eta} & \cdots & w_{2\xi\eta}w_{p\xi\eta} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ w_{p\xi\eta}w_{1\xi\eta} & w_{p\xi\eta}w_{2\xi\eta} & \cdots & w_{p\xi\eta}w_{p\xi\eta} \end{bmatrix} d\xi d\eta \tag{25}$$

$$B = \int_0^1 \int_0^1 M^T M d\xi d\eta = \int_0^1 \int_0^1 \begin{bmatrix} w_1w_1 & w_1w_2 & \cdots & w_1w_p \\ w_2w_1 & w_2w_2 & \cdots & w_2w_p \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ w_pw_1 & w_pw_2 & \cdots & w_pw_p \end{bmatrix} d\xi d\eta \tag{26}$$

For non-trivial solution, the determinant of the matrix H must equal zero. This results in a polynomial equation in λ^2 of degree p, whose solution yields p values of λ^2 from which the corresponding values of the first p natural frequencies can be calculated. λ is called frequency parameter.

2.3 Galerkin Indirect Method

The equation of motion of a thin rectangular plate under flexural free vibration is given by:

$$\frac{\partial^4 W(x,y)}{\partial x^4} + 2\frac{\partial^4 W(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x,y)}{\partial y^4} - \lambda^* W(x,y) = 0 \tag{27}$$

where $\lambda^* = \frac{\rho h \omega^2}{D} = \frac{\lambda^2}{a^4}$ and all other parameters have the connotations as seen in previous sections.

In terms of the non-dimensional coordinates, (27) can be put as:

$$\frac{\partial^4 W(\xi,\eta)}{\partial \xi^4} + 2\alpha^2 \frac{\partial^4 W(\xi,\eta)}{\partial \xi^2 \partial \eta^2} + \alpha^4 \frac{\partial^4 W(\xi,\eta)}{\partial \eta^4} - \lambda^2 W(\xi,\eta) = 0 \tag{28}$$

where $W(\xi,\eta)$ is as given in (15).

Invoking the principle of Galerkin method (Ventsel and Krauthammer, 2001), it follows that:

$$\int_0^1 \int_0^1 \left[\frac{\partial^4 W(\xi,\eta)}{\partial \xi^4} + 2\alpha^2 \frac{\partial^4 W(\xi,\eta)}{\partial \xi^2 \partial \eta^2} + \alpha^4 \frac{\partial^4 W(\xi,\eta)}{\partial \eta^4} - \lambda^2 W(\xi,\eta)\right] \varphi_{ik}(\xi,\eta) d\xi d\eta = 0 \tag{29}$$

where $\varphi_{ik}(\xi,\eta) = F_i(\xi)\, G_k(\eta)$.

(29) yields a system of p = m x n linear algebraic homogeneous equations in the unknown coefficients C_{ik} representing the amplitudes of the vibration modes of the plate. In matrix form, it can be written as:

$$A\,C = 0 \tag{30}$$

where:

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ A_{21} & A_{22} & \dots & A_{2p} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{bmatrix} \tag{31}$$

$A_{11} = \int_0^1 \int_0^1 [L(\varphi_{11})]\varphi_{11} d\xi d\eta\,;\ A_{12} = \int_0^1 \int_0^1 [L(\varphi_{12})]\varphi_{11} d\xi d\eta\,;\dots;\ A_{1n} = \int_0^1 \int_0^1 [L(\varphi_{1n})]\varphi_{11} d\xi d\eta\,;\ A_{1\,(n+1)} = \int_0^1 \int_0^1 [L(\varphi_{21})]\varphi_{11} d\xi d\eta\,;\dots;\ A_{1p} = \int_0^1 \int_0^1 [L(\varphi_{mn})]\varphi_{11} d\xi d\eta\,;\ A_{21} = \int_0^1 \int_0^1 [L(\varphi_{11})]\varphi_{12} d\xi d\eta\,;\dots; A_{pp} = \int_0^1 \int_0^1 [L(\varphi_{mn})]\varphi_{mn} d\xi d\eta$

$L(\dots) = \frac{\partial^4(\dots)}{\partial\xi^4} + 2\alpha^2 \frac{\partial^4(\dots)}{\partial\xi^2\partial\eta^2} + \alpha^4 \frac{\partial^4(\dots)}{\partial\eta^4} - \lambda^2(\dots)$ is the differential operator.

$C = [C_{11}\ \ C_{12}\ \ \dots\ \ C_{1n}\ \ C_{21}\ \ \dots\ \ C_{mn}]^T$, the superscript T refers to matrix transpose.

It follows from (30) that, for non-trivial solution, the determinant of A must equal zero. This will lead to a polynomial equation of degree p in λ^2 whose solution yields p values of the frequency parameter. From these values the approximations of the p consecutive natural frequencies can be computed.

III. RESULTS AND DISCUSSION

3.1 Comparative advantages and disadvantages between Ritz and Galerkin methods

Both Ritz and Galerkin methods rely on the choice of appropriate basis functions in order to yield accurate results. This is where resides their inherent shortcoming. Another disadvantage of both methods is that they are applicable only to simple configurations of plates (rectangular, circular, etc) because of the complexity of selecting the trial functions for domain of complex geometry.

If convenient trial functions are used, one-term approximation gives sufficient accuracy in engineering perspective for both methods. [7, 14]. Both methods have the advantage of yielding approximate results in the form of close form solutions. While the Ritz method requires only that the coordinates functions verify the essential (or kinetic or geometric) boundary conditions, the Galerkin method insists that both the kinetic and natural (or force) boundary conditions must be satisfied by the candidate trial functions. This makes it more difficult to construct trial functions for the Galerkin method than for the Ritz method. Nevertheless, the Ritz method has limited application when compared with the Galerkin method. [7, 15].

3.2 Frequency Parameters of Thin Simply Supported Rectangular Kirchhoff Plates of Various Aspect Ratios

The procedures outlined in sections 2.2 and 2.3 were implemented by means of Mathematica (computer) programmes developed by the authors and making use of the polynomial comparison functions constructed systematically in section 2.1. The programmes were executed for m = n = 1, 2, 3, 4, 5, 6, for various aspect ratios, where m and n are the numbers of trial functions in x and y directions respectively in the shape function. For example, when m and n are kept equal to 1, the programmes yield the approximation of the fundamental frequency parameters; making m = n = 2, 3, 4, 5, 6, the first 4, 9, 16, 25, 36 frequency parameters respectively are obtained. Due to the fact that only a few number (at the lower end) of the Ritz and Galerkin eigenvalues tend to be accurate, only the first six frequency parameters are captured in Table 1. They are compared with exact solutions obtained using the Levy method. The Poisson's ratio, μ was kept equal to 0.3 and the aspect ratios used were 0.4, 0.5, 2/3, 1, 1.5, 2 and 2.5.

It was observed from the data showed in Table 1, that the Ritz and Galerkin methods yielded identical results. This may be explained by the fact that the same coordinate functions were used in the two methods. This confirms the statement of Vendhan and Das [7] which states that the Ritz method and the Galerkin method (based on the direct Euler-Lagrange equations) are mathematically equivalent provided the same set of coordinate functions are used in both methods. The table also shows that, irrespective to the aspect ratio considered, the retained first six frequency parameters are in excellent agreement with the exact solutions: among the six frequency parameters, four gave a null percentage difference while the other two yielded a

percentage difference not exceeding 0.13%. However, the square Kirchhoff plate gave more consistent results as they present a decreasing accuracy with increasing modes of vibration.

Table 1: Comparison of frequency parameters for simply supported rectangular isotropic plates of various aspect ratios with exact solutions

Aspect Ratio		$\lambda = \omega a^2\sqrt{\rho h/D}$ for the Flexural Vibration Modes					
		1	2	3	4	5	6
0.4	Exact solution	11.4487	16.1862	24.0818	35.1358	41.0576	45.795
	Galerkin method	11.4487 (0)*	16.1862 (0)*	24.0856 (0.016)*	35.1752 (0.11)*	41.0576 (0)*	45.795 (0)*
	Ritz method	11.4487 (0)*	16.1862 (0)*	24.0856 (0.016)*	35.1752 (0.11)*	41.0576 (0)*	45.795 (0)*
0.5	Exact solution	12.337	19.7392	32.0762	41.9458	49.348	49.348
	Galerkin method	12.337 (0)*	19.7392 (0)*	32.0827 (0.02)*	41.9458 (0)*	49.348 (0)*	49.4132 (0.13)*
	Ritz method	12.337 (0)*	19.7392 (0)*	32.0827 (0.02)*	41.9458 (0)*	49.348 (0)*	49.4132 (0.13)*
2/3	Exact solution	14.2561	27.4156	43.8649	49.348	57.0244	78.9568
	Galerkin method	14.2561 (0)*	27.4156 (0)*	43.8649 (0)*	49.3606 (0.03)*	57.0244 (0)*	78.9663 (0.01)*
	Ritz method	14.2561 (0)*	27.4156 (0)*	43.8649 (0)*	49.3606 (0.03)*	57.0244 (0)*	78.9663 (0.01)*
1	Exact solution	19.7392	49.348	49.348	78.9568	98.696	98.696
	Galerkin method	19.7392 (0)*	49.348 (0)*	49.348 (0)*	78.9568 (0)*	98.7265 (0.03)*	98.7266 (0.03)*
	Ritz method	19.7392 (0)*	49.348 (0)*	49.348 (0)*	78.9568 (0)*	98.7265 (0.03)*	98.7266 (0.03)*
1.5	Exact solution	32.0762	61.685	98.696	111.033	128.305	177.653
	Galerkin method	32.0762 (0)*	61.685 (0)*	98.696 (0)*	111.061 (0.03)*	128.305 (0)*	177.674 (0.01)*
	Ritz method	32.0762 (0)*	61.685 (0)*	98.696 (0)*	111.061 (0.03)*	128.305 (0)*	177.674 (0.01)*
2	Exact solution	49.348	78.9568	128.305	167.783	197.392	197.392
	Galerkin method	49.348 (0)*	78.9568 (0)*	128.331 (0.02)*	167.783 (0)*	197.392 (0)*	197.653 (0.13)*
	Ritz method	49.348 (0)*	78.9568 (0)*	128.331 (0.02)*	167.783 (0)*	197.392 (0)*	197.653 (0.13)*
2.5	Exact solution	71.5546	101.163	150.511	219.599	256.61	286.219
	Galerkin method	71.5546 (0)*	101.163 (0)*	150.535 (0.02)*	219.845 (0.11)*	256.61 (0)*	286.219 (0)*
	Ritz method	71.5546 (0)*	101.163 (0)*	150.535 (0.02)*	219.845 (0.11)*	256.61 (0)*	286.219 (0)*

* *Percentage difference with respect to exact solution*

Another paramount factor that can be used to measure the efficiency and performance of a computational method is the CPU time elapsed by a computer programme devised for the purpose, to obtain meaningful results. In this study, the developed computer programmes, apart from yielding the frequency parameters, also give the CPU time elapsed to do so. A computer system (with the same specifications) was used to execute the programmes and the different CPU times in seconds were recorded in Table 2. The CPU time differences (which represent the difference between the time elapsed in running the programme used for the Galerkin method and the one elapsed in executing the programme for the Ritz method) were also tabulated according to the number of terms in the shape functions. It was observed from the table that, for a number of terms of 1, 4 and 9 (i.e. when m = n = 1, 2, 3), the CPU time varied from some fractions of second to not up to 9 seconds for the Galerkin method, while the Ritz method CPU time consumption ranged from some fractions of second to not up to 33 seconds, across the aspect ratios considered. In that range, the time differences are negative, meaning that the Galerkin method appeared to be more time-efficient than its counterpart, the Ritz method. Conversely, when the number of terms in the shape function is 16, 25 and 36 (i.e. when m = n = 4, 5 and 6), the CPU time varied from 233.61 seconds up to 1859.36 seconds for the Galerkin method, whereas the

Ritz method took a CPU time ranging from 107.311 seconds to 543.059 seconds, across the aspect ratios considered. In that range, the recorded time differences were all positive and substantial, meaning that the Ritz method is much more time-efficient than the Galerkin method for the higher numbers of terms in the shape functions.

Table 2: CPU times used in Galerkin and Ritz methods to obtain the various frequency parameters

Aspect Ratio		CPU time and time difference in seconds per number of terms in the shape function					
		1 term	4 terms	9 terms	16 terms	25 terms	36 terms
0.4	CPU time (Galerkin Method)	0.063	0.781	7.172	241.312	739.61	1669.21
	CPU time (Ritz Method)	0.109	8.125	32.469	116.218	271.826	543.059
	CPU time difference	-0.046	-7.344	-25.297	125.094	467.784	1126.151
0.5	CPU time (Galerkin Method)	0.047	0.86	8.172	279.485	844.896	1778.26
	CPU time (Ritz Method)	0.125	7.812	30.155	107.311	250.75	493.546
	CPU time difference	-0.078	-6.952	-21.983	172.174	594.146	1284.714
2/3	CPU time (Galerkin Method)	0.031	0.828	7.984	275.125	833.794	1859.36
	CPU time (Ritz Method)	0.125	7.782	31.531	114.827	287.251	528.34
	CPU time difference	-0.094	-6.954	-23.547	160.298	546.543	1331.02
1	CPU time (Galerkin Method)	0.063	0.828	4.718	258.407	773.06	1636.09
	CPU time (Ritz Method)	0.125	7.781	30.375	110.577	259.845	528.35
	CPU time difference	-0.062	-6.953	-25.657	147.83	513.215	1107.74
1.5	CPU time (Galerkin Method)	0.062	0.781	4.765	233.61	740.187	1671.05
	CPU time (Ritz Method)	0.125	8.657	31.234	113.187	266.11	522.23
	CPU time difference	-0.063	-7.876	-26.469	120.423	474.077	1148.82
2	CPU time (Galerkin Method)	0.046	0.828	7.077	241.437	748.983	1646.62
	CPU time (Ritz Method)	0.125	7.907	31.625	114.202	263.608	518.843
	CPU time difference	-0.079	-7.079	-24.548	127.235	485.375	1127.777
2.5	CPU time (Galerkin Method)	0.031	0.812	7.172	256.657	793.827	1823.44
	CPU time (Ritz Method)	0.109	7.906	31.265	110.734	256.749	521.782
	CPU time difference	-0.078	-7.094	-24.093	145.923	537.078	1301.658

IV. CONCLUSION

In this study, systematically constructed comparison functions were used to carry out the free vibration analysis of simply supported rectangular Kirchhoff plates of different aspect ratios, using the Ritz and Galerkin methods. Mathematica computer programmes were developed and used for the purpose. Identical results were obtained using both methods, even though the literature shows that the Galerkin method has a wider application than its counterpart. These results portrayed an excellent agreement with exact solutions, when taken from the

lower end of the eigenvalue spectrum. The results obtained for the square plate were the most consistent. The CPU time efficiency study carried out showed that:

1. For a number of terms in the shape functions of 1, 4 and 9, the Galerkin method proved to be more efficient than the Ritz method.
2. However, for greater numbers of terms in the shape functions, the Ritz method is much more time-efficient than the Galerkin method.

REFERENCES

[1]. Timoshenko, S.P. & Woinowsky-Krieger, S. (1959). *Theory of Plates and Shells*. Second Edition, McGraw-Hill, New York.

[2]. Podio-Guidugli, P. (1989). An Exact Derivation of the Thin Plate Equation. *Journal of Elasticity*, 22: 121 – 133.

[3]. Ventsel, E. & Krauthammer, T. (2001). *Thin Plates and Shells, Theory, Analysis, and Applications*. Marcel Dekker, Inc, New York.

[4]. Ozenda, O. & Virga E. G. (2021). On the Kirchhoff-Love Hypothesis. *Journal of Elasticity*, 143: 359 – 384.

[5]. Eisenberger, M. & Shufrin, I. (2007). The Extended Kantorovich Method for Vibration Analysis of Plates, in Shanmugam, N.E. and Wang, C.M (Ed), *Analysis and Design of Plated Structures*, Woodhead Publishing Ltd., Cambridge, 192-218.

[6]. Leissa, A. W. (1969). *Vibration of Plates*. NASA, Washington D. C.

[7]. Vendhan, C. P. & Das, Y. C. (1975). Application of Rayleigh-Ritz and Galerkin Methods to Non-Linear Vibration of Plates. *Journal of Sound and Vibration*, 39(2), 147 - 157.

[8]. Moreno – Garcia, P., Araujo dos Santos, J. V. & Lopes, H. (2017). A Review and Study on Ritz Method Admissible Functions with Emphasis on Buckling and Free Vibration of Isotropic and Anisotropic Beams and Plates. *Arch Computat Methods Eng*, DOI 10.1007/s11831-017-9214-7

[9]. Singhvi S. & Kapania, R. K. (1994). Comparison of Simple and Chebychev Polynomials in Rayleigh-Ritz Analysis. *J. Eng. Mech.*, 120, 2126-2135.

[10]. Oosterhout, G. M., Van Der Hougt, P. J. M. & Spiering, R. M. E. J. (1995). Accurate Calculation Methods for Natural Frequencies of Plate with Special Attention to the Higher Modes. *Journal of Sound and Vibration*, 183(1), 33-47.

[11]. Monterrubio, L. E. and Ilanko, S. (2012). Sets of Admissible Functions for the Rayleigh-Ritz Method, Proc. 11th International Conf. *on Computational Structures Technology*, Civil-Comp. Press, Stirlingshire, Scotland.

[12]. Dozio, L. (2011). On the Use of Trigonometric Ritz Method for General Vibration Analysis of Rectangular Kirchhoff Plates. *Thin-Walled Structures*, 49, 129 - 144.

[13]. Brown, R. E. & Stone, M. A. (1997). On the use of polynomial series with Rayleigh-Ritz Method. *Composite Structures*, 39(3-4), 191-196.

[14]. Aginam, C. H., Adamou, A. & Nwaiwu C. M. O. (2021). Free Vibration Analysis of all round Clamped Thin Isotropic Rectangular Plate by Ritz Direct Variational Method. *Nigerian Journal of Technology*, Vol. 40, No. 5 (in press).

[15]. Smith, I. M., Farraday, R. V. & O'Connor, B. A. (1973). Rayleigh-Ritz and Galerkin Finite Elements for Diffusion-Convection Problems. *Water Resources Research*, Vol. 9, No. 3, 593 – 606.

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