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He-Laplace Variational Iteration Method for the Analytical Solution of Nonlinear Partial Differential Equations

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Abstract: The goal of this study deals with an effective approach to achieve the approximate solution of nonlinear partial differential equations (NPDE's). This approach exempts the calculation of integration in recurrence relation, and the convolution theorem in Laplace transforms to compute the value of Lagrange multiplier. Finally, nonlinear terms can easily be handled with He's polynomials via homotopy perturbation method (HPM). This approach demonstrates the high efficiency and attains very good agreement in illustrated problems.
Keywords: He-Laplace method, He’s polynomials, KdV equation, Inverse Laplace transform.

I. INTRODUCTION

Most of the applications arising in physical systems are sculpted in NPDE's, that can be broadly useful in numerous disciplines of sciences particularly, fluid mechanics, solid state physics, plasma wave, chemical physics, chemical kinematics, etc. Few years back, both scientists and researchers studied the partial differential equations containing exact and numerical solutions to the nonlinear problems. KdV equations are the mathematical models which may perform a significant role in one-dimensional nonlinear lattice [1]. Numerous approaches have been handled to these problems such as: finite volume scheme [2], spectral method [3], decomposition method [4], (G'/G)-expansion method [5], Reduced differential transformation method [6] and boundary value method [7].

A Chinese mathematician executed an idea of variational iteration method (VIM) to handle the nonlinear hurdles in several sort of NPDE’s, like that: rational solution for KdV equations [8,9]. Another efficient technique, known as homotopy perturbation method [10] has introduced which showed good agreement to solve nonlinear partial differential equations with numerous problems [11]. Many authors [12,13] showed the appropriate solution in various difficulties of NPDE's by merging the Laplace transformations with HPM. It is keenly noticed that the obtained results of Lagrange multiplier are often critical in most of the studies, and all these approaches face some incapacities.

This research article is summarized as follows: In section [II], A new scheme is introduced to handle the NPDE's with some modifications. In section [III], some problems are illustrated to show the capability of the proposed scheme which reveals good results. Finally, conclusion and future work is presented in section[IV].

II. HE-LAPLACE VARIATIONAL ITERATION METHOD

Recently, Nadeem and Li [14,15,16] established a technique to evaluate Lagrange multiplier for ordinary and partial differential equations but we reveal the following modification which present the analytical solution of NPDE’s in a simple way. In order to implement He-LVIM, consider the following differential problem,

$$R(\phi)-S(\phi)-c=0$$

After, taking the Laplace transformation, the recurrence relation may be written as

$$\phi_{n+1}(x,s)=\phi_n(x,s)+\lambda\left\{L\left(R(\phi_n)\right)-L\left(S(\phi_n)+c\right)\right\} \qquad (1)$$

Thus $\lambda(s)$ can be identified by applying the following optimality condition, where, $\tilde{\phi}_n$ is a limited source with

$$\delta\tilde{\phi}_n = 0 \quad \text{and} \quad \frac{\delta\phi_{n+1}(x,s)}{\delta\phi_n(x,s)} = 0 \qquad (2)$$

Thus, inverse Laplace transform of Eq. (1) yields,

$$\phi_{n+1}(x,t) = \phi_n(x,t) + L^{-1}\left[\lambda(s)\left\{L\left(R(\phi_n)\right) - L\left(S(\phi_n)+c\right)\right\}\right]$$

HPM to nonlinear terms can be applied as

$$S(\phi) = \sum_{i=0}^{\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \ldots,$$

where,

$$H_n(\phi_0+\phi_1+\ldots+\phi_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(S\left(\sum_{i=0}^{\infty} p^i \phi_i\right)\right), \quad \text{n=0,1,2,3}$$

So, analogizing the identical, the following approximations are

$$p^0 = \phi_0(x,t) = F(x,t)$$

$$p^1 = \phi_1(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left\{R\phi_0(x,t) + H_0(\phi_0)\right\}\right]$$

$$p^2 = \phi_2(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left\{R\phi_1(x,t) + H_1(\phi_1)\right\}\right]$$

$$p^3 = \phi_3(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left\{R\phi_2(x,t) + H_2(\phi_2)\right\}\right]$$

$$\vdots$$

where,

$$F(x,t) = \phi(x,0) + t\frac{\partial\phi(x,0)}{\partial t}$$

Ultimately, the obtained result provides a series solution

$$\phi(x,t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \ldots,$$

We perform this approach in-depth with the following illustrations.

III. NUMERICAL EXAMPLES

In this part, we will examine a few models to achieve the accuracy and ability of He-LVIM. Significant results demonstrate the efficiency of present method.

Example 1:
Suppose the following fifth-order KdV problem,

$$\phi_t+\phi_x+\phi^2\phi_{xx}+\phi_x\phi_{xx}+20\phi^2\phi_{xxx}+\phi_{xxxxx}=0 \tag{3}$$

with initial condition

$$\phi(x,0)=\frac{1}{x}.$$

Now, applying the Laplace transformation on Eq. (3), we get,

$$L\left(\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}+\phi^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial x}.\frac{\partial^2\phi}{\partial x^2}-20\phi^2\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^5\phi}{\partial x^5}\right)=0.$$

Multiplying this equation with $\lambda_1(s)$, we get,

$$\lambda_1 L\left(\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}+\phi^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial x}.\frac{\partial^2\phi}{\partial x^2}-20\phi^2\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^5\phi}{\partial x^5}\right)=0.$$

According to VIM, the recurrence relation brings the following description,

$$\phi_{n+1}(x,s)=\phi_n(x,s)+\lambda_1 L\left(\frac{\partial\phi}{\partial t}+\frac{\partial\tilde{\phi}}{\partial x}+\phi^2\frac{\partial^2\tilde{\phi}}{\partial x^2}+\frac{\partial\tilde{\phi}}{\partial x}.\frac{\partial^2\tilde{\phi}}{\partial x^2}-20\phi^2\frac{\partial^3\tilde{\phi}}{\partial x^3}+\frac{\partial^5\tilde{\phi}}{\partial x^5}\right) \tag{4}$$

On executing Eq.(2) and taking the variation for solving $\lambda_1(s)$, we get,

$$\lambda_1(s)=-\frac{1}{s}.$$

So, Eq. (4) becomes,

$$\phi_{n+1}(x,s)=\phi_n(x,s)-\frac{1}{s}L\left(\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}+\phi^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial x}.\frac{\partial^2\phi}{\partial x^2}-20\phi^2\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^5\phi}{\partial x^5}\right)$$

Thus, inverse Laplace transform yields,

$$\phi_{n+1}(x,t)=\phi_n(x,t)-L^{-1}\left[\frac{1}{s}L\left(\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}+\phi^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial x}.\frac{\partial^2\phi}{\partial x^2}-20\phi^2\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^5\phi}{\partial x^5}\right)\right]$$

According to He's Polynomial rule and equating the highest power of p, we get,

$$p^0 = \phi_0 = \phi(x,0) = \frac{1}{x}$$

$$p^1 = \phi_1 = -L^{-1}\left[\frac{1}{s}L\left(\frac{\partial\phi_0}{\partial x} + \phi_0{}^2\frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial\phi_0}{\partial x}.\frac{\partial^2\phi_0}{\partial x^2} - 20\phi_0{}^2\frac{\partial^3\phi_0}{\partial x^3} + \frac{\partial^5\phi_0}{\partial x^5}\right)\right] = \frac{t}{x^2}$$

$$p^2 = \phi_2 = -L^{-1}\left[\frac{1}{s}L\left(\frac{\partial\phi_1}{\partial x} + \phi_0{}^2\frac{\partial^2\phi_1}{\partial x^2} + 2\phi_0\phi_1\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial\phi_0}{\partial x}.\frac{\partial^2\phi_1}{\partial x^2} - 20\phi_0{}^2\frac{\partial^3\phi_1}{\partial x^3} - 40\phi_0\phi_1\frac{\partial^3\phi_0}{\partial x^3} + \frac{\partial^5\phi_1}{\partial x^5}\right)\right] = \frac{t^2}{x^3}$$

$$\vdots$$

Hence, the obtained result provides a series solution,

$$\phi(x,t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \ldots,$$

$$= \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \frac{t^3}{x^4}\ldots,$$

Thus, the fifth-order KdV equation has particular solution

$$\phi(x,t) = \frac{1}{x-t}.$$

Example 2:

Next, consider the following K(2,2) problem,

$$\phi_t + (\phi^2)_x + (\phi^2)_{xxx} = 0 \qquad (5)$$

with initial condition

$$\phi(x,0) = x$$

Now, applying the Laplace transformation on Eq. (5)

$$L\left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi^2}{\partial x} + \frac{\partial^3\phi^2}{\partial x^3}\right) = 0.$$

Multiplying this equation with $\lambda_1(s)$, we get,

$$\lambda_2 L\left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi^2}{\partial x} + \frac{\partial^3\phi^2}{\partial x^3}\right) = 0.$$

Thus, the recurrence relation brings the following description,

$$\phi_{n+1}(x,s) = \phi_n(x,s) + \lambda_2 L\left(\frac{\partial\phi}{\partial t} + \frac{\partial\tilde{\phi}^2}{\partial x} + \frac{\partial^3\tilde{\phi}^2}{\partial x^3}\right) = 0. \qquad (6)$$

On executing Eq. (2) and taking the variation for solving $\lambda_1(s)$, we get,

$$\lambda_2(s) = -\frac{1}{s}.$$

So, Eq. (6) becomes,

$$\phi_{n+1}(x,s) = \phi_n(x,s) - \frac{1}{s}L\left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi^2}{\partial x} + \frac{\partial^3\phi^2}{\partial x^3}\right) = 0.$$

Thus, inverse Laplace transform yields,

$$\phi_{n+1}(x,t) = \phi_n(x,t) - L^{-1}\left[\frac{1}{s}L\left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi^2}{\partial x} + \frac{\partial^3\phi^2}{\partial x^3}\right)\right]$$

According to He's Polynomial rule and equating the highest power of p , we get,

$$p^0 = \phi_0 = \phi(x,0) = x$$

$$p^1 = \phi_1 = -L^{-1}\left[\frac{1}{s}L\left(\frac{\partial \phi_0{}^2}{\partial x} + \frac{\partial^3 \phi_0{}^2}{\partial x^3}\right)\right] = -2xt$$

$$p^2 = \phi_2 = -L^{-1}\left[\frac{1}{s}L\left(\frac{\partial\left(2\phi_0\phi_1\right)}{\partial x} + \frac{\partial^3\left(2\phi_0\phi_1\right)}{\partial x^3}\right)\right] = 4xt^2$$

$$p^3 = \phi_3 = -L^{-1}\left[\frac{1}{s}L\left(\frac{\partial\left(\phi_1{}^2 + 2\phi_0\phi_2\right)}{\partial x} + \frac{\partial^3\left(\phi_1{}^2 + 2\phi_0\phi_2\right)}{\partial x^3}\right)\right] = -8xt^3$$

$$\vdots$$

Hence, the obtained result provides a series solution,

$$\begin{aligned}\phi(x,t) &= \phi_0 + \phi_1 + \phi_2 + \phi_3 + \ldots, \\ &= x - 2xt + 4xt^2 - 8xt^3 \ldots,\end{aligned}$$

Thus, the K(2,2) equation has exact solution

$$\phi(x,t) = \frac{x}{1+2t}.$$

IV. CONCLUSION

In this article, He-LVIM has been studied for the approximate solutions of fifth-order KdV and K (2,2) problems. The acquired findings disclose the reliability of the present method and its broader applicability to NPDE's. It is evident that this approach performs immediate successive approximations in need of any constrictive theories or variations. Hence, He-LVIM is reliable and rapidly overcome the difficulties of NPDE's. The present theory reveals the following highlights:

- He-LIVM performed very quickly to the NPDE's.
- He-LVIM is more attractive for solving linear and nonlinear problems with fractal derivatives.
- The suggested method is suitable for fractal derivatives and fractional calculus.
- The computation of Lagrange multiplier by He-LVIM is much easy than variational iteration method.
- All results are computed by Mathematica Software 11.0.1.

In future, He-LVIM must be employed to achieve the estimated solution of fractional PDE's, which are very often used in several branches engineering.

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