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Existence of Mild Solution for Second Order Summation-Difference Equations

S. R. Gadhe, K. L. Bondar
Department of Mathematics, N.W. College, Akhada Balapur, Dist. Hingoli
P.G. Deptartmrnt of Mathematics, NES, Science College, Nanded
Corresponding Author: S. R. Gadhe

Abstract: In this paper, we discuss existence and uniqueness of mild solution of second order initial value problems, with nonlocal conditions, by help of Banach fixed point theorem and the theory of cosine family.

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I. INTRODUCTION

In the present years, the study of difference equations and their applications are found to be more useful in the field of numerical and Engineering as well as social sciences. Agrawal [1], Kelley and Peterson [6], P. Eole (5) had developed theory of difference equation and their inequalities. Some comparison theorems on difference equation and summation equation are obtained by K.L. Bondar etal. [2-4]

Let X be a Banach space with norm ||.|| defined by $\| x \|_b = \sup\{ | x | : x \in B\}, t \in I$ in the closed internal I= [*0,b*]. Let *B=C(J; X)* be Banach space of all continuous functions defined form I into *X*.

Consider the second order nonlinear summation-difference equation with nonlocal conditions:

$$\Delta\left[\Delta x(t-1) + g(t, x(t))\right] = Ax(t) + f\left(t, x(t), \sum_{s=0}^{t-1} k(t,s,x(s))\right), \text{ where } t \in I \qquad (1.1)$$

$$x(0) = x_0 + q(x), \qquad (1.2)$$

$$\Delta x(0) = y_0 + p(x), \qquad (1.3)$$

where A is an infinitesimal small generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space X, $f : I \times X \times X \to X$, $k : I \times I \times X \to X$, $g : I \times X \to X$, q, $p : B \to X$ continuous functions, and x_0, y_0 are elements in *X*

II. PRELIMINARIES AND HYPOTHESES.

In many cases it is advantageous to treat second ordered difference equations directly rather than to convert first order systems. We can study second order equations is the theory of the strongly continuous cosine family.

A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators mapping the Banach space X into itself is called a cosine family if and only if.

(a) *C*(0)= *I* where I is identity operator,

(b) *C(t) x* is strongly continuous in *t* on $\mathbb{R}$ for each fixed $x \in X$;

(c) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

If $\{C(t) : t \in \mathbb{R}\}$ is strongly continuous cosine family in *X*, then $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family, is defined by.

$$S(t)x = \sum_{s=0}^{t-1} C(s)x, \qquad x \in X, t \in \mathbb{R} \qquad (2.1)$$

We define closed operator $G : D(G) \subset X \to X$ It is denoted by $[D(G)]$, the space $[D(G)]$ endowed with the graph norm $\| . \|_G$. The infinitesimal small generator $A : X \to X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$Ax = \Delta^2 C(t)x|_{t=0}$, $x \in D(A)$,

where D(*A*) = { $x \in X : C(.)x \in C^2$ (ℝ, *X*)}. Moreover, $M \geq 1$ and *N* are positive constants such that $\| C(t) \| \leq M$ and $\| S(t) \| \leq N$ for every $t \in I$.

If *x* (.) is a solution of (1.1)-(1.3), g is *D(A)*-valued and $Ag \in B$ then

$$x(t) = C(t)[x_0 + q(x)] + S(t)[y_0 + p(x) + g(0, x_0 + q(x))]$$
$$-\sum_{s=0}^{t-1} g(s, x(s)) - \sum_{s=0}^{t-1} AS(t-s) \sum_{r=0}^{s-1} g(\tau, x(\tau))$$
$$+\sum_{s=0}^{t-1} S(t-s) f(s, x(s), \sum_{\tau=0}^{t-1} k(s,\tau,x(\tau)), \; t \in I. \qquad (2.2)$$

The expression (2.2) and the relation A $\sum_{\theta=s}^{t-1} S(\theta)x = C(t)x - C(s)x$, are the motivation of the following definition.

Definition 2.1 *A function x* ∈ *B is a mild solution of the abstract nonlocal Cauchy problem,* (1.1)-(1.3) *if condition* (1.2) *is verified and satisfy*

$$x(t) = C(t)[x_0 + q(x)] + S(t)[y_0 + p(x) + g(0, x_0 + q(x))]$$
$$-\sum_{s=0}^{t-1} C(t-s) g(s, x(s)) + \sum_{s=0}^{t-1} S(t-s)$$
$$\times f\left(f(s, x(s), \sum_{\tau=0}^{s-1} k(s,\tau,x(\tau)), \right) \qquad t \in I. \qquad (2.3)$$

We list the following hypotheses for our convenience.

(H_1) *X* is a Banach space with norm ||.|| and $x_0, y_0 \in X$

(H_2) $t \in I$ and $B_r = \{z : \| z \| \leq r\} \subset X$.

(H3) f: $I \times X \times X \to X$ satisfies the following conditions:

(a) The function $f(t,.,.) : X \times X \to X$ is continuous a.e. $t \in I$.

(b) The function $f(., x, y) : I \to X$ is s measurable for each $(x, y) \in X \times X$,

(c) There exists a positive constant L_f such that

$\| f(t, x_1, y_1) - f(t, x_2, y_2) \| \leq L_f (\| x_1 - x_2 \| + \| y_1 - y_2 \|)$, for $t \in I$, $x_i, y_i \in X$.

(H4) the function $g : I \times X \to X$ satisfies the following conditions:

(d) The function $g(t,.) : X \to X$ is continuous a.e. $t \in I$.

(e) The function $g(,.x) : X \to X$ is measurable for each $x \in X$.

(f) There exists a positive constant L_g such that

$\| (g(t, x_1) - g(t, x_2) \| \leq L_g \| x_1 - x_2 \|)$, for $t \in I$, $x_i \in X$.

(g) There exist positive constants and such that

$\| g(t, x) \| \leq c_1 \| x \| + c_2$ for $t \in I$, $x_i, y_i \in X$.

(H_5) $k : I \times I \times X \to X$ is continuous in *t*, *s* on *I* and there exists a positive constant *K* such that $\| k(t, s, x_1) - k(t, s, x_2) \| \leq K \| x_1 - x_2 \|$, for $0 \leq x \leq t \in I, x_i \in X$.

(H_6) $q, p : B \to X$ are continuous and there exist constants $L_q, L_p > 0$ such that

$\| q(x_1) - q(x_2) \| \leq L_q \| x_1 - x_2 \|_b$ and $\| p(x_1) - p(x_2) \| \leq L_p \| x_1 - x_2 \|_b$, for $x_1, x_2 \in C(I, B_r)$.

(H_7) A is the infinitesimal generator of strongly continuous cosine family{*C(t)*: *t* ∈ ℝ }and *S(t)* the sine function associated with *C*(t) which is defined in (2.1).

(H_8) $L_1 = \max_{t \in J} \| f(t, 0, 0) \|$, $K_1 = \max_{t \in J} \| k(t, s, 0) \|, Q = \max_{x \in C(J, B_r)} \| q(x) \|$,

$P = \max_{x \in C(J,B_r)} \| p(x) \|$.

(H$_9$) The constants $\|x_0\|$, $\|y_0\|, c_1, c_2, r, b, L_f, L_g, K, L_q, L_p, M, N, L_1, K_1, Q,$ and P satisfy the following inequalities:

$$M[\| x_0 \| + Q] + N[\| y_0 \| + P + c_1(\| x_0 \| + Q) + c_2] + M(c_1 r + c_2)b$$
$$+ N\left(L_f rb + L_f Krb^2 + L_f K_1 b^2 + L_1 b\right) \le r,$$

and

$$\left[M\left(L_q + L_g b\right) + N\left(L_p + L_g L_q + L_f b + L_f Kb^2\right)\right] < 1.$$

Now, we establish our result of existence.

III. EXISTENCE OF MILD SOLUTION

Theorem 3.1 *Suppose that the hypotheses* [H$_1$] – [H$_9$] *hold, then the problem* (1.1)-(1.3) *has a unique mild solution on interval I*

Proof: Let $Z = C(I, B_r)$ and define an operator $F : Z \to Z$ by

$$(Fz)(t) = C(t)[x_0 + q(z)] + S(t)[y_0 + p(z) + g(0, x_0 + q(z))]$$

$$-\sum_{s=0}^{t-1} C(t-s) g(s, z(s)) + \sum_{s=0}^{t-1} S(t-s) \times f\left(f(s, z(s), \sum_{\tau=0}^{t-1} k(s, \tau, z(\tau)), \right) \quad (2.4)$$

$t \in J$.

We firstly show that F maps Z into itself. By applying the operator F in (2.4) and above mention hypotheses then

$$\| (Fz)(t) \| \le \| C(t)[x_0 + q(z) \| + \| S(t)[y_0 + p(z) + g(0, x_0 + q(z))] \|$$
$$+ \sum_{s=0}^{t-1} \| C(t-s) g(s, z(s)) \| + \sum_{s=0}^{t-1} \| S(t-s) f\left(s, z(s), \sum_{\tau=0}^{t-1} k(s, \tau, z(\tau)) \right) \|$$
$$\le M[\|x_b\| + Q] + N[\|y_0\| + P \|g(0), x_0 + q(z)\|] + M \sum_{s=0}^{t-1} \| g(s, z(s) \|$$
$$+ N \sum_{s=0}^{t-1} \left[\| f(s, z(s), \sum_{\tau=0}^{s-1} k(s, \tau, z(\tau))) - f(s, 0, 0) + f(s, 0, 0) \| \right]$$
$$\le M\left[\| x_0 \| + Q\right] + N\left[\| y_0 \| + P + c_1(\| x_0 \| + \| Q \|) + c_2\right] + M \sum_{s=0}^{t-1} (c_1 r + c_2)$$
$$+ N \sum_{s=0}^{t-1} \left[L_f \left(r + \sum_{\tau=0}^{s-1} \| kr + k_1 b) + L_1 \right) \right]$$
$$\le M\left[\| x_0 \| + Q\right] + N\left[\| y_0 \| + P + c_1(\| x_0 \| + Q) + c_2\right] + M(c_1 r + c_2)b$$
$$+ N\left[L_f rb + L_f Krb^2 + L_f K_1 b^2 + L_1 b\right]$$
$$\le r, \quad (3.1)$$

for $z \in Z$ and $t \in I$. Hence $F(Z) \subset Z$. Therefore, the equation (3.1) shows that the operator F maps Z into itself.

Now, we shall show that F is a contraction on Z. For every $z_1, z_2 \in Z$ and $t \in I$, we obtain.

$$\| (Fz_1)(t) - (Fz_2)(t) \| \le \| C(t) \| \| q(z_1) - q(z_2) \|$$
$$+ \| S(t) \| \left[\| p(z_1) - p(z_2) \| + \| g(0, x_0 + q(z_1)) - g(0, x_0 + q(z_2)) \| \right]$$
$$+ \sum_{s=0}^{t-1} \| C(t-s) \left[g(s, z_1(s)) - g(s, z_2(s)) \right] \|$$

$$+\sum_{s=0}^{t-1}\| S(t-s)\left[f\left(s, z_1(s), \sum_{\tau=0}^{s-1} k(s,\tau,z_1(\tau)\right) - f(s, z_2(s)), \sum_{\tau=0}^{s-1} k(s,\tau,z_2(\tau))\right]\|$$

$$+L_g + \| q(z_1) - q(z_2) \|_b \, b + ML_g \| z_1 - z_2 \|_b \, b$$

$$+N\sum_{s=0}^{t-1} L_f \left[\| z_1 - z_2 \|_b + K \| z_1 - z_2 \|_b\right]$$

$$\leq \left[M\left(L_q + L_g b\right) + N\left(L_p + L_g L_q + L_f b + L_f K b^2\right)\right] \| z_1 - z_2 \| b$$

If we consider $l = \left[M\left(L_q + L_g b\right) + N\left(L_p + L_g L_q + L_f b + L_f K b^2\right)\right]$, then

$$\| Fz_1 - Fz_2 \|_b \leq l \| z_1 - z_2 \|_b$$

with *0 < l <1.* Thus F is a contraction on the complete metric space Z, therefore by the Banach fixed point theorem the function F has a unique fixed point in the space Z and this point is the mild solution of problem (1.1)-(1.3) on J.

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