Intuitionistic Fuzzy Hilbert Space And Some Properties

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Abstract: In this paper we introduce Intuitionistic Fuzzy Hilbert space (IFHS) and investigate certain some properties of IFHS. Also, we have given several definitions of Intuitionistic Fuzzy Orthogonal and Intuitionistic Fuzzy Orthonormal discuss in detail.

Key Words: IntuitionisticFuzzy Norm(IFN), Intuitionistic Fuzzy Inner Product Space(IFIP-space), Intuitionistic Fuzzy Hilbert Space(IFH-space), Intuitionistic Fuzzy Orthogonal(IF-orthogonal), Intuitionistic Fuzzy Orthonormal(IF-, Orthonormal).

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I. INTRODUCTION


In section 2, some definitions and preliminary results are given which are used in this paper. In section 3 we introduced the concept of intuitionistic fuzzy Hilbert space (IFH-space), intuitionistic fuzzy orthogonal, intuitionistic fuzzy orthonormal and its some properties.

II. PRELIMINARIES

Definition (2.1): [15]
Let U be a linear space over the real or complex field F. Let N and M be two fuzzy subsets of U x R such that 
\[ \forall x, y \in U \text{ and } c \in F \text{ such that } \]
\[ (I1) \quad N(x, t) + M(x, t) \leq 1 \]
\[ (I2) \quad (\forall t > 0, N(x, t) = 1 ) \text{ iff } x = 0 \]
\[ (I3) \quad (\forall t \in R \text{ and } t > 0, N(cx, t) = N(x, \frac{t}{|c|}), \text{ if } c \neq 0 \]
\[ (I4) \quad (\forall s, t \in R, x, u \in U, N(x + y, s + t) \geq N(x, s) \land N(y, t) \]
\[ (I5) \quad N(x, \ast) \text{ is non-decreasing function of } R \text{ and } \lim_{t \to \infty} N(x, t) = 1 \]
\[ (I6) \quad (\forall t \leq 0, M(x, t) = 1 \]
\[ (I7) \quad (\forall t > 0, M(x, t) = 0 ) \text{ iff } x = 0 \]
\[ (I8) \quad (\forall t \in R \text{ and } t > 0, M(cx, t) = M(x, \frac{t}{|c|}), \text{ if } c \neq 0 \]
\[ (I9) \quad (\forall s, t \in R, x, u \in U, M(x + y, s + t) \leq M(x, s) \lor M(y, t) \]
\[ (I10) \quad M(x, \ast) \text{ is non-increasing function of } R \]

The pair (N, M) is called an intuitionistic fuzzy norm on U and the triplet (U, N, M) is called an intuitionistic fuzzy normed linear space (IFNLS).

Note 2.1:[15]
(N,M) is an intuitionistic fuzzy norm on U in the sense that associated to \( x \in U \) and \( t \in R \), \( N(x,t) \) indicates the grade of the statement, “the norm of \( x \leq t \) “, and which belongs to \([0,1]\), whereas \( M(x,t) \) indicates the grade of the statement, “the norm of \( x \leq t \) “and which also belongs to \([0,1]\).

**Theorem 2.1:** [15]

Let \((N,M)\) be an intuitionistic fuzzy norm on a linear space U. Assume that the following property holds:

\[(M(x,t) \leq N(x,t)) \Rightarrow x = 0\]

Define \( \|x\|^N = \{t \geq 0 : N(x,t) \geq t\} \) and \( \|x\|^M = \{t > 0 : M(x,t) \leq 1 - \alpha\}, \alpha \in (0,1). \) Then \( \{(\|\cdot\|^N, \|\cdot\|^M): \alpha \in (0,1)\} \) forms a family of \( \alpha \)-ascending pair of norms on linear space U satisfying the inclusion condition: for each \( \alpha \in (0,1) \), \( \|x\|^N \geq \|x\|^M. \)

**Theorem 2.2:** [15]

Let \( \{(\|\cdot\|^N, \|\cdot\|^M): \alpha \in (0,1)\} \) be a family of a \( \alpha \)-pair of ascending norms on a linear space U satisfying the inclusion condition.

Let \( N: U \times R \rightarrow I = [0,1] \) be defined as follows:

\[
N(x,t) = \begin{cases} 
0 & \text{when } (x,t) \neq (0,0) \\
\alpha & \text{when } (x,t) = (0,0)
\end{cases}
\]

Then \((N,M)\) is an intuitionistic fuzzy norm on U.

**Definition 2.2:** [15]

Let \((U, N, M)\) be an IFNL-space and \( \alpha \in (0,1) \). A sequence \( \{x_n\} \) in \( U \) is said to be \( \alpha \)-convergent sequence in U if \( \exists x \in U \), such that \( \lim_{n \rightarrow \infty} N(x_n - x, t) > \alpha \) \( \forall t > 0 \) and \( \lim_{n \rightarrow \infty} M(x_n - x, t) \leq 1 - \alpha, t > 0 \), and \( x \) is called the limit of \( \{x_n\} \).

**Proposition 2.1:** [15]

Let \((U, N, M)\) be an IFNL-space satisfying (Theorem 2.1). If \( \{x_n\} \) be an \( \alpha \)-convergent sequence in \((U, N, M)\) then \( \|x_n - x\|^N \rightarrow 0 \) as \( n \rightarrow \infty \) and \( \|x_n - x\|^M \rightarrow 0 \) as \( n \rightarrow \infty \). \( \|\cdot\|^N \) denotes \( \alpha \)-norm of \( N \), and \( \|\cdot\|^M \) denotes \( \alpha \)-norm of \( M \).

**Definition 3.2:** [16] **Intuitionistic Fuzzy Inner Product Space (IFIP-space)**

Let \( V \) be a linear space and let \( \vartheta: V \times V \rightarrow [0,1] \) be two mappings such that the following holds \( \forall x, y \in V \) and \( s, t \in C \).

\[(IFIP1) \mu(x,y,t) = \vartheta(x,y,t) \leq 1 \]

\[(IFIP2) \mu(x+y,z,t+|s|) \geq \mu(x,z,|t|) \land \mu(y,z,|s|) \]

\[(IFIP3) \mu(x,y,|st|) \geq \mu(x,|s|,|t|) \land \mu(y,|s|,|t|) \]

\[(IFIP4) \mu(x,y,t) = \mu(y,x,t) \]

\[(IFIP5) \mu(ax,y,t) = \mu(x,y,\frac{1}{a^t}) \text{ if } a(\neq 0) \in C \]

\[(IFIP6) \mu(x,x,t) = 1 \forall t > 0 \text{ if } x = 0 \]

\[(IFIP7) \mu(x,x,t) = \begin{cases} 
1 & \text{if } t > 0 \\
0 & \text{if } t = 0
\end{cases} \]

\[(IFIP8) \vartheta(x+y,z,t+|s|) \leq \vartheta(x,z,|t|) \lor \vartheta(y,z,|s|) \]

\[(IFIP9) \vartheta(x,y,z,t+|st|) \leq \vartheta(x,y,|st|) \lor \vartheta(y,x,|st|) \]

\[(IFIP10) \vartheta(x,y,t) = \vartheta(y,x,t) \]

\[(IFIP11) \vartheta(ax,y,t) = \vartheta(x,y,\frac{1}{a^t}) \text{ if } a(\neq 0) \in C \]

\[(IFIP12) \forall \vartheta(t \in C \setminus R^+, \vartheta(x,x,t) = 1 \]

\[(IFIP13) \vartheta(x,x,t) = \vartheta(x,x,t) \rightarrow [0,1] \text{ is a monotonic not increasing function and } \vartheta(x,y,t) = 0 \]

Then the pair \((\mu, \vartheta)\) will be called an intuitionistic fuzzy inner product space(IFIP-space) and we shall call the triple \((V, \mu, \vartheta)\) an intuitionistic fuzzy inner product space.

**Note 2.2:** [16]

\((\mu, \vartheta)\) is an intuitionistic fuzzy inner product on \( V \) in the sense that associated to \( x, y \in V \) and \( t \in C \), \( \mu(x,y,t) \) is grade of statement, “the inner product of \( x \) and \( y \) is equal to \( t \)”, and which belongs to \([0,1]\), where as \( \vartheta(x,y,t) \) indicates the grade of statement, “the inner product of \( x \) and \( y \) is not equal to \( t \)” and which also belongs to \([0,1]\).

**Note 2.3:** [16]

We may observe that the property:

\[\mu(x,x,t) = 0 \forall t \leq 0 \text{ and } x \in V \]

holds from (IFIP1) and (IFIP12). Similarly from (IFIP1) and (IFIP6) implies \((\vartheta(x,x,t) = 0) \forall t > 0 \) \( \Leftrightarrow x = 0 \).

**Theorem 2.3:** [16]

Let \((\mu, \vartheta)\) be an IFIP on V. Then if we define two functions \( N, M: V \times R \rightarrow I = [0,1] \) as follows:
(3.2): (IFHS)

Let \((V, \mu, \theta)\) be an IFIP space. \(V\) is said to be IFHS if it is \(l\) – Complete.

Definition (3.3): \((\alpha - \text{IF Orthogonal})\)

Let \(\alpha \in (0,1)\) and \((V, \mu, \theta)\) be an IFIP space satisfying Remark 2.1(IFIPS 14,14a,15,15a). Now if \(x, y \in V\) be such that \((x,y)_\alpha^N = 0\) and \((x,y)_\alpha^M = 0\), then we say that \(x, y\) are \(\alpha\) – IF orthogonal to each other and it is denoted by \(x \perp_\alpha^N y\) and \(x \perp_\alpha^M y\).

Let \(A\) be a subset of \(V\) and \(x \in V\). Now if \((x,y)_\alpha^N = 0\ \forall y \in A\) and \((x,y)_\alpha^M = 0\ \forall y \in A\), then we say that \(x\) is \(\alpha\) – IF orthogonal to \(A\). It is denoted by \(x \perp_\alpha^N A\) and \(x \perp_\alpha^M A\).

Definition (3.4): (IF Orthogonal)

Let \((V, \mu, \theta)\) be an IFIP space satisfying Remark 2.1(IFIPS 14,14a,15,15a). Now if \(x, y \in V\) be such that \((x,y)_\alpha^N = 0\) and \((x,y)_\alpha^M = 0\), \(\forall \alpha \in (0,1)\), then we say that \(x, y\) are intuitionistic fuzzy orthogonal to each other and it is denoted by \(x \perp y\). Thus \(x \perp y\) if \(x \perp_\alpha^N y\) and \(x \perp_\alpha^M y\ \forall \alpha \in (0,1)\).

Definition (3.5): \((\alpha - \text{IF Orthonormal})\)

Let \((V, \mu, \theta)\) be an IFIP space satisfying Remark 2.1(IFIPS 14,14a,15,15a) and \(\alpha \in (0,1)\). An \(\alpha\) – IF orthogonal set \(A\) in \(V\) is said to be \(\alpha\) – IF orthonormal if the elements have \(\alpha\) – norm \(l\) that is \(\forall x, y \in A\),

\[
(x,y)_\alpha^N = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}
\]

N(x,t) = \[\mu(x,t^2) \forall t \in R \text{ and } t > 0 \]
M(x,t) = \[\theta(x,t^2) \forall t \in R \text{ and } t > 0 \]

Then \((N,M)\) is an intuitionistic fuzzy norm on \(V\) of the type given in [6].

Proposition 2.2: [16]

If \((\mu_1, \theta_1)\) and \((\mu_2, \theta_2)\) are two-intuitionistic fuzzy inner product on \(V\), then \((\mu, \theta) = (\mu_1 \wedge \mu_2, \theta_1 \vee \theta_2)\) is also, an IFIP on \(V\).

Proposition 2.3: [16]

Let \((V,\mu, \theta)\) be an IFIP space then

(i) For \(x, y, z \in V \text{ and } t \in C, \mu(x,y + z, |t| + |s|) \geq \mu(x,y, |t|) \wedge \mu(x,z, |s|) \text{ and } \theta(x,y + z, |t| + |s|) \leq \theta(x,y, |t|) \vee \theta(x,z, |s|)

(ii) For \(x, y, z \in V \text{ and } t \in C, \mu(x,x, t^2) \geq \alpha \Rightarrow \theta(x,x, t^2) \leq 1 - \alpha

Remark 2.1: [16]

1. Assume that \(\theta\) satisfies the following condition(IFIP14) \(\theta((x,x,t^2) < 1 \forall t > 0) \Rightarrow x = 0\) then from(IFIP1) and(IFIP14), it follows that(IFIP14a) \(\mu(x,x,t^2) > 0 \forall t > 0 \Rightarrow x = 0\).

2. By the theorem 2.3 and decomposition theorem 1.1 and 1.2 we have the following:

Let \((V, \mu, \theta)\) be an IFIP space satisfying (IFIP14). Define \(\forall \alpha \in (0,1), \|x\|_\alpha^N = \Lambda(t > 0; \mu(x,x,t^2) \geq \alpha \text{ and } xA = A t > 0; 2x,t \leq \alpha \text{ and } \theta(x,x, t^2) \leq 1 - \alpha\) then \(\alpha = 0,1\) is an ascending family of pair of crisp norms on \(V\) generated from \((\mu, \theta)\) and which satisfies the inclusion condition(IC): Because, \(\forall t > 0, \mu(x,x,t^2) \geq \alpha \Rightarrow \theta(x,x, t^2) \leq 1 - \alpha

In the sequel we shall consider the following conditions:

(IFIP15) \(\forall x, y \in V \text{ and } p, q \in R, \mu(x+y, x+y, 2q^2) \Lambda \mu(x-y, x-y, 2p^2) \geq \mu(x,x, p^2) \Lambda \mu(y,y, q^2)\)

and (IFIP15a) \(\theta(x+y, x+y, 2q^2) \vee \theta(x-x, x-x, 2p^2) \leq \theta(x,x, p^2) \vee \theta(y,y, q^2)\)

Proposition 2.4: [16] (Parallelogram law)

Let \((\mu, \theta)\) be an IFIP space on a linear space \(V\) over the field of complex numbers satisfying (IFIP1), (IFIP15) and (IFIP15a).

Let \(\alpha \in (0,1)\) and \(\|x\|_\alpha^N\) be the \(\alpha\) norms generated from IFIP(\(\mu\), \(\theta\)) on \(V\). Then

\[\|x - y\|_\alpha^N^2 + \|x + y\|_\alpha^N^2 = 2([\|x\|_\alpha^N^2 + \|y\|_\alpha^N^2] \text{ and } \|x - y\|_\alpha^M^2 + \|x + y\|_\alpha^M^2 = 2([\|x\|_\alpha^M^2 + \|y\|_\alpha^M^2])\]
where $\langle \cdot, \cdot \rangle_\alpha^N$ and $\langle \cdot, \cdot \rangle_\alpha^M$ are the induced inner product by $(\mu, \theta)$.

**Definition (3.6): (IF Orthonormal)**

Let $(V, \mu, \theta)$ be an IFIPS space satisfying Remark 2.1(IFIPS 14.14a,15.15a). A IF orthogonal set $A$ in $V$ is said to be IF orthonormal, if the elements have $\alpha$- norm $1 \forall \alpha \in (0,1)$ that is $\forall x, y \in A$,

$$
\langle x, y \rangle^N_\alpha = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y
\end{cases} \quad \& \quad \langle x, y \rangle^M_\alpha = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y
\end{cases}
$$

where $\langle \cdot, \cdot \rangle_\alpha^N$ and $\langle \cdot, \cdot \rangle_\alpha^M$ are the induced inner product by $(\mu, \theta)$.

**Note 3.1:**

An $\alpha$ - IF orthonormal set and IF orthonormal set in an IFIPS space are linearly independent.

**Proposition (3.1):**

Let $(e_k)$ be an IF orthonormal sequence in an IFHS $(V, \mu, \theta)$ satisfying Remark 2.1(IFIPS 14.14a,15.15a). Then the series $\sum_1^\infty \alpha_k e_k$ converges (w.r.t. $\| \cdot \|_\alpha^N$, $\| \cdot \|_\alpha^M$; $\alpha \in (0,1)$, where $\| \cdot \|_\alpha^N$ and $\| \cdot \|_\alpha^M$ are the $\alpha$ - norms of $(N, M)$ which are induced by $(\mu, \theta)$) iff $\sum_k |\alpha_k|^2$ converges w.r.t. $\| \cdot \|_\alpha^N$ and $\| \cdot \|_\alpha^M$.

**Proof:**

Let $S_n = \sum_{k=1}^n \alpha_k e_k$ and $S_n = \sum_{k=1}^n |\alpha_k|^2$. Then

$$
\| S_n - S_m \|_\alpha^N = \sum_{k=m+1}^n |\alpha_k|^2
$$

and

$$
\| S_n - S_m \|_\alpha^M = \sum_{k=m+1}^n |\alpha_k|^2
$$

i.e. $\| S_n - S_m \|_\alpha^N = |\alpha_{m+1}|^2 + |\alpha_{m+2}|^2 + \cdots + |\alpha_n|^2$

and

$$
\| S_n - S_m \|_\alpha^M = |\alpha_{m+1}|^2 + |\alpha_{m+2}|^2 + \cdots + |\alpha_n|^2
$$

i.e. $\| S_n - S_m \|_\alpha^M = \sigma_n - \sigma_m \& \| S_n - S_m \|_\alpha^M = \sigma_n - \sigma_m \quad \forall \alpha \in (0,1)$

Hence $S_n$ is a Cauchy sequence w.r.t. $\| \cdot \|_\alpha^N$ and $\| \cdot \|_\alpha^M$ w.r.t $\forall \alpha \in (0,1)$ iff $\sigma_n$ is Cauchy in $R$(set of all real numbers). Hence $S_n$ is Cauchy iff $\sigma_n$ is Cauchy in $R$.

**Note 3.2:**

In case of $\alpha$ – IF orthonormal sequence the proposition holds.

**Proposition (3.2):**

Let $(V, \mu, \theta)$ be an IFHS satisfying Remark 2.1(IFIPS 14.14a,15.15a) and $\alpha \in (0,1)$ and let $(e_k)$ be an $\alpha$ - IF orthonormal sequence in $V$. If the series $\sum_1^\infty \sum_1^\infty \beta_k e_k$ and $\sum_1^{\infty} \beta_k^\ast e_k$ is $\alpha$-convergent w.r.t. $(N, M)$ induced by $(\mu, \theta)$, then the coefficients $\beta_k = \langle x, e_k \rangle^N_\alpha$ and $\beta_k^\ast = \langle x, e_k \rangle^M_\alpha$, denotes the sums $\sum_1^\infty \beta_k e_k \& \sum_1^\infty \beta_k^\ast e_k$ and hence $x = \sum_1^\infty \beta_k e_k \& x = \sum_1^\infty \beta_k^\ast e_k$.

**Proof:**

Since $\sum_1^\infty \sum_1^\infty \beta_k e_k \& \sum_1^\infty \beta_k^\ast e_k$ are $\alpha$-convergent w.r.t. $\| \cdot \|^N_\alpha \& \| \cdot \|^M_\alpha$ (def 2.2 & Prop2.1)

$$
\| S_n - S_m \|^N_\alpha \\& \| S_n - S_m \|^M_\alpha \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \& \quad \| S_n - S_m \|^N_\alpha \\& \| S_n - S_m \|^M_\alpha \rightarrow 0 \quad \text{as } n \rightarrow \infty
$$

Let $S_n = \sum_1^\infty \beta_k e_k$ and $S_n = \sum_1^\infty \beta_k^\ast e_k$.

Taking IP with $S_n$ and $S_n$ and using the definition of $\alpha$ – IF orthogonality, we have

$$
\langle S_n, e_j \rangle^N_\alpha = \beta_j \\& \langle S_n, e_j \rangle^M_\alpha = \beta_j^\ast \quad \text{for } j = 1, 2, \ldots, k.
$$

Hence the proof.

**Theorem (3.1):**

Let $(V, \mu, \theta)$ be an IFHS satisfying Remark 2.1(IFIPS 14.14a,15.15a) and let $(e_k)$ be an IF orthonormal sequence in $V$. If the series $\sum_1^\infty \gamma_k e_k$ and $\sum_1^\infty \gamma_k^\ast e_k$ converge w.r.t. $(N, M)$ induced by $(\mu, \theta)$, then

$$
\gamma_k = \langle x, e_k \rangle^N_\alpha \\& \gamma_k^\ast = \langle x, e_k \rangle^M_\alpha \quad \forall \alpha, \beta \in (0,1)
$$

Where $\langle \cdot, \cdot \rangle^N_\alpha$ and $\langle \cdot, \cdot \rangle^M_\alpha$ denote $\alpha$- IP induced by $(\mu, \theta)$, $x$ denotes the sum of $\sum_1^\infty \gamma_k e_k \& \sum_1^\infty \gamma_k^\ast e_k$.

Hence $x = \sum_1^\infty \gamma_k e_k \\& x = \sum_1^\infty \gamma_k^\ast e_k$.

**Proof:**

Since $(e_k)$ is IF orthonormal, it is orthonormal w.r.t. each $\langle \cdot, \cdot \rangle^N_\alpha$ and $\langle \cdot, \cdot \rangle^M_\alpha$, $\alpha \in (0,1)$.

Now $\sum_1^\infty \gamma_k e_k$ and $\sum_1^\infty \gamma_k^\ast e_k$ is convergent w.r.t. $(N, M)$ implies they are convergent w.r.t. $\| \cdot \|^N_\alpha$ and $\| \cdot \|^M_\alpha$, $\alpha \in (0,1)$ [by Proposition(3.1)]. We have
\[ \langle \sum_{k=1}^{n} y_k e_k, e_j \rangle = y_j = \langle \sum_{k=1}^{n} y_k e_k, e_j \rangle' \quad \forall j \neq j' \in \mathbb{N} \]

Taking limit as \( n \to \infty \), we have

\[
\lim_{n \to \infty} \langle \sum_{k=1}^{n} y_k e_k, e_j \rangle = \lim_{n \to \infty} y_j = \langle \sum_{k=1}^{n} y_k e_k, e_j \rangle' = \lim_{n \to \infty} \langle \sum_{k=1}^{n} y_k e_k, e_j \rangle'
\]

Since \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle' \) are continuous \( \forall \alpha \in (0,1) \)

\[
\langle x, e_j \rangle = \langle x, e_j \rangle' \quad \forall \alpha, \beta \in (0,1), \quad j = 1,2,3, \ldots
\]

Hence the proof.

**Definition (3.7): (Complete IF Orthonormal Set)**

Let \( (V, \mu, \theta) \) be an IFIP orthonormal set satisfying Remark 2.1(IFIPS 14,14a,15a). An IF orthonormal set \( A \subset V \) is called complete IF orthonormal set if there is no \( \alpha \) – IF orthonormal set (\( \alpha \in (0,1) \)) of which A is a proper subset. If \( \alpha \) is countable then it is called complete IF orthonormal sequence.

**Theorem (3.2): (Bessel’s Inequality)**

Let \( (V, \mu, \theta) \) be an IFIP space satisfying Remark 2.1(IFIPS 14,14a,15a), \( \alpha \in (0,1) \) and \( \{ e_i \} \) be an \( \alpha \) – IF orthonormal sequence in \( V \). Then for every \( x \in V \),

\[
\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \| x \|_{\alpha}^2 \quad \forall \alpha \in (0,1)
\]

**Proof:**

Since \( \alpha \) – IF orthonormal sequence of \( \alpha \) – IF orthonormal sequence in \( (V, \langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle') \) so by Bessel’s inequality in crisp inner product we have

\[
\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \| x \|_{\alpha}^2 \quad \forall \alpha \in (0,1)
\]

**Theorem (3.3):**

Let \( (V, \mu, \theta) \) be a Hilbert space satisfying Remark 2.1(IFIPS 15,15a) and \( \{ e_i \} \) is an IF orthonormal sequence in \( V \) then the following statements are equivalent.

(i) \( \{ e_i \} \) is complete IF orthonormal.

(ii) For every \( x \in V \), \( x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \) \& \( x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \) \( \forall \alpha \in (0,1) \) and hence

\[
\langle x, e_i \rangle' = \langle x, e_i \rangle \quad \forall \alpha, \beta \in (0,1), \quad \text{i.e.} \ x \text{ is independent on } \alpha.
\]

(iii) For every \( x \in V \), \( \| x \|_{\alpha}^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \) \& \( \| x \|_{\alpha}^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \) \( \forall \alpha \in (0,1) \)

\[
\text{and hence } \| x \|_{\alpha}^2 = \| x \|_{\alpha}^2 \quad \forall \alpha \in (0,1)
\]

**Proof:**

(a) Suppose (i) holds.

Let \( \{ e_i \} \) be a complete IF orthonormal sequence and \( x \perp e_i \) for \( i = 1,2, \ldots \)

\( \Rightarrow x \perp x \wedge e_i \quad \forall \alpha \in (0,1), \quad i = 1,2, \ldots \)

\( \Rightarrow \langle x, e_i \rangle = 0 \) \& \( \langle x, e_i \rangle' = 0 \quad \forall \alpha \in (0,1) \)

Set for fixed \( \alpha_0 \), \( e_0 = \frac{x}{\| x \|_{\alpha_0}} \)

\[
\text{Then } |\langle e_0, e_0 \rangle|_{\alpha_0}^2 = \langle e_0, e_0 \rangle_{\alpha_0}^2 = 1 \quad \text{& } |\langle e_0, e_i \rangle|_{\alpha_0}^2 = 0 \quad \text{for } i = 1,2, \ldots
\]

\( \Rightarrow e_0 = \frac{x}{\| x \|_{\alpha_0}} \).

\( \text{Then } |\langle e_0, e_0 \rangle|_{\alpha_0}^2 = \langle e_0, e_0 \rangle_{\alpha_0}^2 = 1 \quad \text{& } |\langle e_0, e_i \rangle|_{\alpha_0}^2 = 0 \quad \text{for } i = 1,2, \ldots
\)

\( \therefore \) we get an \( \alpha_0 \) – IF orthonormal sequence \( \{ e_0, e_1, e_2, \ldots \} \) of which \( \{ e_1, e_2, \ldots \} \) is proper subset – a contraction to completeness.

\( \Rightarrow e_0 = 0 \Rightarrow x = 0 \)

So (i) \( \Rightarrow \) (ii).

(b) Suppose (ii) holds.

Let \( x \perp e_i \) for \( i = 1,2, \ldots \) \( \Rightarrow x = 0 \).

\( \Rightarrow x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \perp e_i \quad \forall \alpha \in (0,1) \)

\( \Rightarrow x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \perp e_i \quad \forall \alpha \in (0,1) \)

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\[ \Rightarrow x - \sum_{i=1}^{\infty} (x, e_i)^{N} e_i = 0 \quad \& \quad x - \sum_{i=1}^{\infty} (x, e_i)^{M} e_i = 0 \]

\[ \Rightarrow x = \sum_{i=1}^{\infty} (x, e_i)^{N} e_i = \sum_{i=1}^{\infty} (x, e_i)^{M} e_i \quad \& \quad x = \sum_{i=1}^{\infty} (x, e_i)^{N} e_i = \sum_{i=1}^{\infty} (x, e_i)^{M} e_i \quad [ \text{since } x = 0 ] \]

\[ \Rightarrow \sum_{i=1}^{\infty} ((x, e_i)^{N} - (x, e_i)^{P}) e_i = 0 \quad \& \quad \sum_{i=1}^{\infty} ((x, e_i)^{M} - (x, e_i)^{P}) e_i = 0 \]

Since \{e_i\} is linearly independent, therefore
\[ (x, e_i)^{N} - (x, e_i)^{P} = 0 \quad \& \quad (x, e_i)^{M} - (x, e_i)^{P} = 0 \quad \forall \alpha, \beta \in (0, 1) \]

\[ \Rightarrow (x, e_i)^{N} = (x, e_i)^{M} \quad \& \quad (x, e_i)^{M} = (x, e_i)^{P} \]

Thus (ii) \(\Rightarrow\) (iii).

(c) Suppose (iii) holds.

Let \[ x = \sum_{i=1}^{\infty} (x, e_i)^{N} e_i \quad \& \quad x = \sum_{i=1}^{\infty} (x, e_i)^{M} e_i \quad \forall \alpha \in (0, 1) \]

Now \[ \|x\|^{N} = (x, x)^{N} \|x\|^{M} = (x, x)^{M} \]

\[ \Rightarrow \|x\|^{N} = \sum_{i=1}^{\infty} (x, e_i)^{N} e_i \|x\|^{M} = \sum_{i=1}^{\infty} (x, e_i)^{M} e_i \]

\[ \Rightarrow \lim_{n \to \infty} \|x\|^{N} = \lim_{n \to \infty} \sum_{i=1}^{n} (x, e_i)^{N} e_i \|x\|^{M} = \lim_{n \to \infty} \sum_{i=1}^{n} (x, e_i)^{M} e_i \]

\[ \Rightarrow \|x\|^{N} = \lim_{n \to \infty} \sum_{i=1}^{n} (x, e_i)^{N} e_i \|x\|^{M} = \lim_{n \to \infty} \sum_{i=1}^{n} (x, e_i)^{M} e_i \]

\[ \Rightarrow \|x\|^{N} = \sum_{i=1}^{\infty} ((x, e_i)^{N})^{2} \|x\|^{M} = \sum_{i=1}^{\infty} ((x, e_i)^{M})^{2} \]

Now from (iii) we have \[ (x, e_i)^{N} = (x, e_i)^{M} \quad \& \quad (x, e_i)^{M} = (x, e_i)^{P} \quad \forall i = 1, 2, 3, \ldots \quad \forall \alpha, \beta \in (0, 1) \]

Using this we get
\[ \Rightarrow \|x\|^{N} = \sum_{i=1}^{\infty} (x, e_i)^{N} \|x\|^{M} = \sum_{i=1}^{\infty} (x, e_i)^{M} \|x\|^{P} \]

\[ i.e. \quad \|x\|^{N} = \|x\|^{P} \quad \text{and} \quad \|x\|^{M} = \|x\|^{P} \quad \forall \alpha, \beta \in (0, 1) \]

So (iii) \(\Rightarrow\) (iv).

(d) Suppose (iv) holds \{e_i\} is not complete. Then we get for an \(\alpha \in (0, 1)\) \(e^{\alpha} , e_1, e_2, \ldots \) of which \{e_1, e_2, \ldots \} is proper subset and \(\|e^{\alpha}\|^{N} = 1, \|e^{\alpha}\|^{M} = 1, \|e^{\alpha}\|^{P} = 0 \forall i = 1, 2, 3, \ldots \)

\[ \|e^{\alpha}\|^{N} = \sum_{i=1}^{\infty} (e^{\alpha}, e_i)^{N} = 0 \quad \& \quad \|e^{\alpha}\|^{M} = \sum_{i=1}^{\infty} (e^{\alpha}, e_i)^{M} = 0 \]

Thus (iv) \(\Rightarrow\) (i).

### III. CONCLUSION

From this paper, the idea of Intuitionistic Fuzzy Hilbert space (IFHS) is relatively new. We attempted to prove some properties of Intuitionistic Fuzzy Hilbert space and some important concepts viz, definitions of intuitionistic fuzzy orthogonal, intuitionistic fuzzy orthonormal etc., The results of this paper will be helpful for researchers to develop Intuitionistic fuzzy functional analysis.

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### REFERENCES


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