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Fuzzy Normal Operator in Fuzzy Hilbert Space and its Properties

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Abstract: In this work we focus our study on Fuzzy Normal operator acting on a Fuzzy Hilbert space (FH-space).we have given several definitions, theorems and discuss in detail the properties of Fuzzy Normal operator in FH-space.
Keywords: Adjoint Fuzzy operator, FH-space,Fuzzy Normal operator,Self-Adjoint Fuzzy operator.

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I. INTRODUCTION

In 1984 Katsaras[3]first introduced the notation of Fuzzy Norm on a linear space then after any other mathematicians like Felbin[5],Cheng and Mordeson[12], S.K.Samanta[15]etc.,have been taken a definition of fuzzy normed spaces. The definition of fuzzy innerproduct space(FIP-space) was headmost start by R.Biswas[10] and after that according the chronological in[6],[7],[11],[4],[9].Modulate the definition of fuzzy inner product space(FIP-space) has been insert by M.Goudarzi and S.M.Vaezpour in[8],[14]. Also, in [8]and [13]given the connotation of a Fuzzy Hilbert space(FH-Space).In 2017, sudad M.Rasheed defined concept of adjoint fuzzy linear operators, self- adjoint fuzzy linear operators.

In this paper we consider a self-adjoint operator in FH-space and introduce the definition of Fuzzy Normaloperator, we establish a theorem from a fuzzy normal operator in FH-space
The organization of this paper is as follows:

In section 2 provides some preliminaryresults, which are used in this paper.In section 3 we introduce the idea of Fuzzy Normal operators, severaltheorems, discuss properties of such fuzzy operators.

II. PRELIMINARIES

Definition (2.1): [8]
A fuzzy inner product space (FIP-Space) is a triplet (X,F,*),where X is a real vector space, * is a continuous t-norm, Fis a fuzzy set on $X^2\times \mathbb{R}$ satisfying the following conditions for every x,y,z$\in X$ and s,r,t$\in R$.
FI-1:F(x, x, 0)=0 and F(x,x,t)>0, for each t>0

FI-2:F(x,x,t)$\neq H(t)$ for some t$\in R$ if and only if x$\neq$ 0,where $H(t) = \begin{cases} 1, & if t > 0 \\ 0, & if t \le 0 \end{cases}$

FI-3:F(x,y,t)= F(y,x,t)

FI-4: For any $\alpha \in \mathbb{R}$,F (αx, y, t)= $\begin{cases} F\left(x, y, \frac{t}{\alpha}\right) \alpha > 0 \\ H(t) \alpha = 0 \\ 1 - F\left(x, y, \frac{t}{-\alpha}\right) \alpha < 0 \end{cases}$

FI-5:F(x,x,t)*F(y,y,s)≤F(x+y,x+y,t+s)
FI-6:$\mathrm{Sup}_{s+r=t}$ [F(x,z,s)* F(y,z,r)]= F(x+y,z,t)
FI-7:F(x,y,.):$\mathbb{R} \rightarrow [0,1] is continuous on \mathbb{R}\backslash\{0\}$
FI-8:$\lim_{t\to\infty} F(x, y, t) = 1$

Definition (2.2): [14]
Let (E, F,*) be probabilistic inner product space.
1. A sequence $\{x_n\}\in E$ is called $\mathcal{T}$ - converges to x$\in E$,if for any $\epsilon > 0$ and λ>0,∃N∈ Z^+,
N= N (ϵ, λ) such that Fx_n-x,x_n-x (ϵ)>1- λ whenever n>N.

2. A linear Functional f(x) defined on E is called $\mathcal{T}_F$ -continuous, if $x_n \xrightarrow{\mathcal{T}_F} x$ implies $f(x_n) \xrightarrow{\mathcal{T}_F} f(x)$ for any $\{x_n\}$, x$\in E$

Theorem (2.3): [8]
Let (X,F,*) be a FIP- space,where * is strong t- norm, and for each x $,y \in X$ sup {t $\in \mathbb{R}$:F(x,y,t)<1}<∞.Define$\langle .,. \rangle$:X×X→ $\mathbb{R}$ by $\langle x,y \rangle = Sup$ {t∈ $\mathbb{R}$:F(x,y,t)<1}. Then (X,$\langle .,. \rangle$) is a IP- space (inner product space), so that (X,$\|.\|$)is a N-space (normed space), where$\|.\|=\langle x,x \rangle^{1/2}$, $\forall x \in X$.
Definition(2.4):[8]
Let (X, F,*) be a FIP- space with IP $\langle x,y \rangle = Sup$ {t∈ $\mathbb{R}$:F(x,y,t)<1},$\forall x, y \in X$. If X is complete in the $\|.\|$, then X is called Fuzzy Hilbert – space (FH-space).
Theorem(2.5):[8]
Let(X,F,*) be a FH- space with IP $\langle x,y \rangle = Sup$ {t∈ $\mathbb{R}$:F(x,y,t)<1},$\forall x, y \in X$ for $x_n \in X$ and $x_n \xrightarrow{\|.\|} x$, then $x_n \xrightarrow{\mathcal{T}_F} x$.
Theorem (2.6): (Rise theorem) [8], [14]
Let(X, F,*) be FH- space. For any$\mathcal{T}_F$ –continuousfunctional,∃unique $y \in X$such that for all$x \in X$, we have g(x)= Sup {t∈ $\mathbb{R}$:F(x,y,t)<1}.
Theorem (2.7): [1]
Let (E,G,*) be a FIP space, where * is strong t- norm, and sup {x ∈ $\mathbb{R}$:G(u,v,t)<1}< ∞ for allu,v ∈ E,then Sup{x∈ $\mathbb{R}$:G(u+v,w,x)<1}= Sup{x∈ $\mathbb{R}$:G(u,w,x)<1}+ Sup {x∈ $\mathbb{R}$:G(v,w,x)<1}$\forall u, v, w \in E$.
Remark (2.8): [1]
Let FB (E) the set of all fuzzy linear operators on E.
Theorem (2.9): [1](Adjoint Fuzzy operator in FH-space)
Let (E, G,*) be a FH-space, Let S∈ $FB(E)$ be $\mathcal{T}_F$ –continuous linear functional, then ∃ unique S^*∈ $FB(E)$ such that $\langle Su,v \rangle = \langle u,S^*v \rangle$ }$\forall u, v \in E$.
Definition (2.10): [1]
Let (E,G,*) be a FH-space with IP: $\langle u,v \rangle$ = Sup {x∈ $\mathbb{R}$:G(u,v,x)<1},$\forall u, v \in E$ and let S ∈ $FB(E)$,then S is self –adjoint Fuzzy operator. If S=S^* where S* is adjoint Fuzzy operator of S.
Theorem (2.11): [1]
Let (E,G,*) be a FH-space with IP: $\langle u,v \rangle = Sup$ {x∈ $\mathbb{R}$:G(u,v,x)<1} and let S ∈ $FB(E)$,then S isself –adjoint Fuzzy operator.
Theorem (2.12):[1]
Let (E,G,*) be a FH-space with IP: $\langle u,v \rangle = Sup$ {x∈ $\mathbb{R}$:G(u,v,x)<1},$\forall u, v \in E$ and let S^* be the adjoint Fuzzy operator of S∈ $FB(E)$, then:

i. $(S^*)^*=S$
ii. $(\alpha S)^*=\alpha S^*$
iii. $(\alpha S+\beta T)^*= \alpha S^* + \beta T^*$ where α,β are scalars and T∈ $FB(E)$.
iv. $(ST)^*=T^*S^*$

Theorem (2.13): [1]
Let (E,G,*) be a FH-space with IP: $\langle u,v \rangle$ = Sup {x∈ $\mathbb{R}$:G(u,v,x)<1}, $\forall u, v \in E and$let S ∈ $FB(E)$ then,$\|Su\| = \|S^*u\|$ for all u∈ E.

III. MAIN RESULTS

In this section we introduce the definition of fuzzy normal operator in FH-space as well as some elementary properties of fuzzy normal operator in FH-space are presented.

Definition (3.1): Fuzzy Normal Operator
Let $(X, F,*)$ be a FH-space with IP:$\langle u, v \rangle = \sup\{x \in R: F(u, v, x) < 1\} \forall u, v \in X$ and let$S \in FB(X)$. Then S is a fuzzy normal operator if it commutes with its (fuzzy) adjoint i.e$SS^* = S^*S$.
Note:

1. In the triplet $(X, F,*)$ where 'X' is a real vector space, '$*$' is a continuous t-norm and 'F' is a fuzzy set on $X^2 \times \mathbb{R}$.
2. $FB(X)$ means the set of all fuzzy continuous(bounded) linear operators.

Remark:

1. It is obvious that every self-adjoint fuzzy operator is fuzzy normal.
2. If S is fuzzy normal and α is a scalar then 'αS' is also a fuzzy normal.
3. The limit S of any fuzzy convergent sequence $\{S_k\}$ of fuzzy normal operator is fuzzy normal.

We know that ${S_k}^* \to S_k$ so

$$\|SS^* - S^*S\| \le \|SS^* - S_k{S_k}^*\| + \|S_k{S_k}^* - {S_k}^*S_k\| + \|{S_k}^*S_k - S^*S\|$$
$$= \|SS^* - S_k{S_k}^*\| + \|{S_k}^*S_k - S^*S\|$$
$$\to 0$$

Implies that $SS^* = S^*S$.

Theorem (3.2):

If S_1 and S_2 are fuzzy normal operators on $(X, F,*)$ with the property that either commutes with fuzzy adjoint of the other, then $S_1 + S_2$ and S_1S_2 are fuzzy normal.

Proof:

It is clear by taking fuzzy adjoints that $S_1S_2{}^* = S_2{}^*S_1$ iff $S_2S_1{}^* = S_1{}^*S_2$.

So, the assumption implies that each commute with fuzzy adjoint of the other.

(i) To show that $S_1 + S_2$ is fuzzy normal under the stated conditions we have only to compare the results of the following computations.

$$(S_1 + S_2)(S_1 + S_2)^* = (S_1 + S_2)(S_1{}^* + S_2{}^*)$$
$$= S_1S_1{}^* + S_1S_2{}^* + S_2S_1{}^* + S_2S_2{}^* \qquad \text{----------(1)}$$

& $$(S_1 + S_2)^*(S_1 + S_2) = (S_1{}^* + S_2{}^*)(S_1 + S_2)$$
$$= S_1{}^*S_1 + S_1{}^*S_2 + S_2{}^*S_1 + S_2{}^*S_2 \qquad \text{----------(2)}$$

From (1) and (2)

Now $(S_1 + S_2)(S_1 + S_2)^* = (S_1 + S_2)^*(S_1 + S_2)$

Thus $S_1 + S_2$ is fuzzy normal.

(ii) To show that S_1S_2 is fuzzy normal:

$$S_1S_2(S_1S_2)^* = S_1S_2S_2{}^*S_1{}^*$$
$$= S_1S_2{}^*S_2S_1{}^*$$
$$= S_2{}^*S_1S_1{}^*S_2$$
$$= S_2{}^*S_1{}^*S_1S_2$$
$$= (S_1S_2)^*S_1S_2$$

Now $S_1S_2(S_1S_2)^* = (S_1S_2)^*S_1S_2$

Thus S_1S_2 is fuzzy normal.

Theorem (3.3):

Let $(X, F,*)$ be a FH-space with IP:$\langle u, v\rangle = \sup\{x \in R: F(u, v, x) < 1\}$and let $S \in FB(X)$ be a fuzzy normal operator iff$\|S^*u\| = \|Su\| \forall u \in X$.

Proof:

Let $\|S^*u\| = \|Su\|$

$\Leftrightarrow \|S^*u\|^2 = \|Su\|^2$

$\Leftrightarrow \langle S^*u, S^*u\rangle = \langle Su, Su\rangle =$

$\Leftrightarrow Sup\{x \in R: F(S^*u, S^*u, x) < 1\} = Sup\{x \in R: F(Su, Su, x) < 1\}$

$\Leftrightarrow Sup\{x \in R: F(SS^*u, u, x) < 1\} = Sup\{x \in R: F(S^*Su, u, x) < 1\}$

$\Leftrightarrow \langle SS^*u, u\rangle = \langle S^*Su, u\rangle$

$\Leftrightarrow \langle SS^*u - S^*Su, u\rangle = 0$

$\Leftrightarrow \langle (SS^* - S^*S)u, u\rangle = 0$

$\Leftrightarrow (SS^* - S^*S)u = 0$

$\Leftrightarrow SS^* - S^*S = 0$

$\Leftrightarrow SS^* = S^*S$

Hence Proved.

Theorem (3.4):

Let $(X, F,*)$ be a FH-space with IP:$\langle u, v\rangle = \sup\{x \in R: F(u, v, x) < 1\}$and let $S \in FB(X)$ be a fuzzy normal operator then $\|S^2\| = \|S\|^2$.

Proof:

Let $\|S^2u\|^2 = \|SSu\|^2$

$= \langle SSu, SSu\rangle$

$= Sup\{x \in R: F(SSu, SSu, x) < 1\}$

$= Sup\{x \in R: F(S^*Su, S^*Su, x) < 1\}$

$= \langle S^*Su, S^*Su\rangle$

$= \|S^*Su\|^2$

i.e$\|S^2u\|^2 = \|S^*Su\|^2$

Which implies $\|S^2u\| = \|S^*Su\|$

$\|S^2\| = \|S^*S\| \forall u \in X$

By known result

$\|S^*S\| = \|S\|^2$

Therefore, we get

$\|S^2\| = \|S\|^2$

Remark:

Complex number z can be expressed uniquely represented as $z = a + ib$, where a and b are real numbers. And these real numbers are called real and imaginary parts of z.

$$a = \frac{z+\bar{z}}{2} \& b = \frac{z-\bar{z}}{2i}$$

The analogy between general operators and complex numbers and between self-adjoint fuzzy operators and real numbers suggests that for an arbitrary operator $S \in FB(X)$,we form,

$$T_1 = \frac{S+S^*}{2} \& T_2 = \frac{S-S^*}{2i}$$

$T_1 \& T_2$ are clearly fuzzy self-adjoint we have the property that

$$S = T_1 + iT_2$$

The uniqueness of this expression for S follows at once

$$S^* = T_1 - iT_2$$

The self-adjoint operators $T_1 \& T_2$ are called real part and imaginary part of S.

Theorem (3.5):

Let $(X, F, *)$ be a FH-space with IP:$\langle u, v\rangle = \sup\{x \in R: F(u, v, x) < 1\}$and let $S \in FB(X)$ be a fuzzy normal operator then S is fuzzy normal iff its real and imaginary parts commute.

Proof:

If T_1 and T_2 are real and imaginary parts of S. So that

$$S = T_1 + iT_2 \quad \& \quad S^* = T_1 - iT_2$$

Then

$$SS^* = (T_1 + iT_2)(T_1 - iT_2)$$

$$= {T_1}^2 + {T_2}^2 + i(T_2T_1 - T_1T_2) \qquad \text{------------(3)}$$

$$S^*S = (T_1 - iT_2)(T_1 + iT_2)$$

$$= {T_1}^2 + {T_2}^2 + i(T_1T_2 - T_2T_1) \qquad \text{------------(4)}$$

It is clear that if $T_1T_2 = T_2T_1$,

Then from (3) & (4),

$$SS^* = S^*S$$

Conversely,

If $SS^* = S^*S$, then $T_1T_2 - T_2T_1 = T_2T_1 - T_1T_2$

So $2T_1T_2 = 2T_2T_1$

Implies that $T_1T_2 = T_2T_1$

Hence proved.

Problem (3.6):

Let $(X, F, *)$ be a FH-space with IP: $\langle u, v\rangle = \sup\{x \in R: F(u, v, x) < 1\}$and let $S \in FB(X)$ be an arbitrary (fuzzy) operator and if α and β are scalars such that $|\alpha| = |\beta|$ show that $\alpha S + \beta S^*$ is normal.

Solution:

From theorem (3.3) it is enough to prove $\|(\alpha S + \beta S^*)^* u\| = \|(\alpha S + \beta S^*) u\|$

Let $\|(\alpha S + \beta S^*)^* u\|^2 = \langle(\alpha S + \beta S^*)^* u, (\alpha S + \beta S^*)^* u\rangle$

$=\langle(\alpha S^* + \beta (S^*)^*)u, (\alpha S^* + \beta (S^*)^*)u\rangle$

$= \langle(\alpha S^* + \beta S)u, (\alpha S^* + \beta S)u\rangle$

$= Sup\{x \in R: F\big((\alpha S^* + \beta S)u, (\alpha S^* + \beta S)u, x\big) < 1\}$

$= Sup\{x \in R: F(\alpha S^* u, \alpha S^* u, x) < 1\} +$

$Sup\{x \in R: F(\beta S u, \beta S u, x) < 1\}$

$= Sup\{x \in R: F(\alpha S u, \alpha S u, x) < 1\} +$

$Sup\{x \in R: F(\beta S^* u, \beta S^* u, x) < 1\}$ [since $S^* = S$]

$= Sup\{x \in R: F\big((\alpha S + \beta S^*)u, (\alpha S + \beta S^*)u, x\big) < 1\}$

$= \langle(\alpha S + \beta S^*)u, (\alpha S + \beta S^*)u\rangle$

$= \|(\alpha S + \beta S^*)u\|^2$

i.e.$\|(\alpha S + \beta S^*)^* u\|^2 = \|(\alpha S + \beta S^*)u\|^2$

$$\|(\alpha S + \beta S^*)^* u\| = \|(\alpha S + \beta S^*)u\|$$

There fore $\alpha S + \beta S^*$ is normal.

IV. CONCLUSION

From this paper, the idea of fuzzy normal operator in FH-space is relatively new. We attempted to prove some properties of fuzzy normal operator in fuzzy Hilbert space. The results of this paper will be helpful for researchers to develop fuzzy functional analysis.

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