A New Approach To Solve Fuzzy Fractional Assignment Problem By Using Taylor’s Series Method

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Abstract – In this paper, a method of solving the fuzzy fractional assignment problems, where the cost of the objective function is expressed as triangular fuzzy number, is proposed. In the proposed method, the linear programming of fuzzy fractional assignment problem (FFAP) is transformed to a multi objective linear programming (MOLP) of assignment problem and resultant problem is converted to a linear programming problem, using Taylor’s series method. An illustrative numerical example is given to justify the proposed theory.

Keywords – Fuzzy Fractional Assignment Problem, Triangular Fuzzy Number, Taylor Series Approach

Date of Submission: 11-05-2018

I. INTRODUCTION

An assignment problem is a special type of linear programming problem where the objective is to assign a number of persons to n jobs at a minimum cost (time). Zadeh [18] introduced the concept of fuzzy sets to deal with imprecision, vagueness in real life situations. Since, then tremendous efforts have been spent, significant advances have been made on the development of numerous methodologies and applications to various decision problems. Fuzzy assignment problems have received great attention in recent years. The term AP (Assignment Problem) first appeared in Votaw and Orden [16]. Hungarian method by Kuhn [10] is widely used for the solution to AP’s. Chen [4] proposed a fuzzy assignment model that did not consider the differences of individuals and also proved some theorems. Wang [17] solved a similar model by graph theory. Dubois and Fortemps [7] proposed a flexible AP which combines with fuzzy theory, multiple criteria decision making and constraint- directed methodology. Lin and Wen [11] investigated a fuzzy AP in which the cost depends on the quality of the job.

But in various applications such as production planning, financial planning etc. the decision maker may be interested in optimizing an objective function having ratio of linear function. These types of problems can be handled by using linear fractional programming problem (LFPP) techniques. The field of linear fractional programming problem was developed by Hungarian mathematician B. Matros in 1960 [12]. The linear fractional programming problem is an important planning tool for the past decades which is applied to different discipline engineering, business, finance and economics. Linear fractional problem is generally used for modelling real life problems with one or more objectives such as profit cost, inventory sales, actual cost-standard cost and so forth. M. Shigeno et. al [15] proposed polynomial time algorithm for fractional assignment problem. In this approach the fractional assignment problem is interpreted as bipartite graph with vertex sets and edge sets. Later on several authors such as Bajalinov [1], Charnes and Cooper [3], Odior [13], Panday and Punnen [14] proposed different approaches for solving LFPP. M. Shigeno et. al [15] proposed polynomial time algorithm for fractional assignment problem. In this approach the fractional assignment problem is interpreted as bipartite graph with vertex sets and edge sets.

D.M. Doke et al. [5,6] developed an algorithm to solve multi objective fractional programming by Taylor series approach and fuzzy compromise approach. Further they also developed an algorithm to solve multi objective linear fractional transportation problem in which each objective function is expanded about optimal solution by Taylor Series method.

Guzel N. and Sivri M. [8] proposed a solution to multi objective linear fractional programming problem by expanding the objective function as 1st order Taylor series at the optimal points. These multi objective linear programming problem is reduced to single objective problem by assuming the weight of each objective function is equal and considering the sum of these linear objective functions. N. Cetin and F. Tiryaki [2] proposed a fuzzy approach to generate a compromise pareto optimal solution for multi objective fractional transportation problem by reducing the problem to the Zimmermann’s ‘min’ operator model and constructing a solution based on generalized Dinkelbach’s algorithm.
II. PRELIMINARIES

In this section, we briefly review some basic concepts of fuzzy numbers.

2.1 Basic Definitions

In this section, some basic definitions are reviewed.

Definition 2.1 [9] The characteristic function \( \mu_A \) of a crisp set \( A \subseteq X \) assigns a value either 0 or 1 to each member in \( X \). This function can be generalized to a function \( \mu_A^\mu \) such that the value assigned to the element of the universal set \( X \) fall within a specified range i.e. \( \mu_A^\mu : X \to [0,1] \). The assigned value indicate the membership grade of the element in the set \( A \).

The function \( \mu_A^\mu \) is called the membership function and the set \( \{ (x, \mu_A^\mu(x)) ; x \in X \} \) defined by \( \mu_A^\mu(x) \) for each \( x \in X \) is called fuzzy set.

Definition 2.2 [9] A fuzzy set \( \tilde{A} \) defined on the universal set of real numbers \( R \), is said to be a fuzzy number if its membership function has following characteristics:

(i) \( \tilde{A} \) is a convex i.e. \( \mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \quad \forall x_1, x_2 \in R, \lambda \in [0,1] \).

(ii) \( \tilde{A} \) is normal i.e. \( \exists x_0 \in R \) such that \( \mu_{\tilde{A}}(x_0) = 1 \).

(iii) \( \mu_{\tilde{A}} \) is piecewise continuous.

Definition 2.3 [9] A fuzzy number \( \tilde{A} \) defined on the universal set of real numbers \( R \), denoted as \( \tilde{A} = (a, b, c) \), is said to be a triangular fuzzy number if its membership function, \( \mu_{\tilde{A}}(x) \), is given by

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{(x-a)}{(b-a)}, & a \leq x < b \\
1, & x = b \\
\frac{(x-c)}{(b-c)}, & b < x \leq c \\
0, & \text{otherwise}
\end{cases}
\]

2.2 Arithmetic Operation of Triangular Fuzzy Number

The following are the four operations that can be performed on triangular fuzzy numbers [9]:

Let \( \tilde{A} = (a_1, a_2, a_3) \) and \( \tilde{B} = (b_1, b_2, b_3) \) then,

(i) Addition: \( \tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \)

(ii) Subtraction: \( \tilde{A} - \tilde{B} = (a_1 - b_1, a_2 - b_2, a_3 - b_3) \).

(iii) Multiplication: \( \tilde{A} \times \tilde{B} = (\min(a_1b_1, a_2b_2, a_3b_3), a_2b_3, \max(a_1b_1, a_2b_2, a_3b_3)) \)

(iv) Division: \( \tilde{A} / \tilde{B} = (\min(a_1/b_1, a_1/b_3, a_2/b_1, a_2/b_3, a_3/b_1, a_3/b_3), a_2/b_2, \max(a_1/b_1, a_1/b_3, a_2/b_1, a_2/b_3, a_3/b_1, a_3/b_3)) \).

III. FUZZY ASSIGNMENT PROBLEM

Suppose there are ‘n’ jobs to be performed and ‘n’ persons are available for doing these jobs. Assume that each person can do one job at a time and each job can be assigned to one person only. Let \( \tilde{C}_{ij} \) be the appropriate cost if \( i^{th} \) job is assigned to \( j^{th} \) person. The problem is to find an assignment \( x_{ij} \) (which job should be assigned to which person) so that the approximate total cost for performing all jobs is minimum. The above problem may be formulated as follows

\[
\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{C}_{ij} x_{ij}
\]

Subject to
\[ \sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, 3, \ldots, n \]
\[ \sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, 3, \ldots, n \]
\[ x_{ij} = 0 \text{ or } 1 \quad \forall \ i, j \]

**IV. FUZZY FRACTIONAL ASSIGNMENT PROBLEM**

The general formation of fuzzy fractional assignment problem may be written as:

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij} \]
\[ \text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{d}_{ij} x_{ij} \]

Subject to
\[ \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, n \]
\[ \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n \]
\[ x_{ij} = 0 \text{ or } 1 \quad \forall \ i, j \]

where \( \tilde{c}_{ij} \) and \( \tilde{d}_{ij} \) are represented as fuzzy number.

**V. PROPOSED METHOD FOR SOLVING FFAP BY USING TAYLOR’S SERIES APPROACH.**

The steps of proposed method are as follows:

**Step 1** Suppose there are ‘n’ persons and ‘n’ jobs. Each job must be done by exactly one person, also each person can do, at most, one job. The problem is to assign the people to the jobs so as to minimize the total cost of completing all of the jobs. The general FFAP can be mathematically formulated as:

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} (c_1, c_2, c_3) x_{ij} \]
\[ \text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} (d_1, d_2, d_3) x_{ij} \]

Subject to
\[ \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, n \]
\[ \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n \]
\[ x_{ij} = 0 \text{ or } 1 \quad \forall \ i, j \]
where \( \tilde{c}_{ij} = (c_1, c_2, c_3) \) and \( \tilde{d}_{ij} = (d_1, d_2, d_3) \) are represented as triangular fuzzy numbers.

Now the problem can be reduced as follows:

\[
\text{Min } Z = \left( \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} c_{1} x_{ij}, \sum_{i=1}^{n} \sum_{j=1}^{n} c_{2} x_{ij}, \sum_{i=1}^{n} \sum_{j=1}^{n} c_{3} x_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} d_{1} x_{ij}, \sum_{i=1}^{n} \sum_{j=1}^{n} d_{2} x_{ij}, \sum_{i=1}^{n} \sum_{j=1}^{n} d_{3} x_{ij}} \right)
\]

Subject to

\[
\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, ..., n
\]

\[
\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, ..., n
\]

\( x_{ij} = 0 \) or \( 1 \) \( \forall \ i, j \)

**Step 2** The problem, obtained in Step 1, can be reduced as the following deterministic multi objective fractional assignment problem.

\[
\text{Min } Z_1 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} c_{1} x_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} d_{3} x_{ij}}
\]

\[
\text{Min } Z_2 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} c_{2} x_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} d_{2} x_{ij}}
\]

\[
\text{Min } Z_3 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} c_{3} x_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} d_{1} x_{ij}}
\]

subject to

\[
\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, ..., n
\]

\[
\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, ..., n
\]
Step 3 The problem obtained in Step 2, can be considered as three Linear fractional assignment problems with objective function \( Z_k \), \( k = 1,2,3 \) respectively, with the optimal solution \( Z_k^* \) of each problem can be obtained using Charnes and Cooper’s linear transform technique [3].

Step 4 Determine \( x_{ij}^* \) which is the value that is used to minimize the \( k^{th} \) objective function \( Z_k(x) \) where \( k = 1, 2, 3 \) by using obtained \( Z_k^* \), \( k = 1,2,3 \) and the Taylor series approach.

The multi objective fractional assignment problem can be converted into a single objective linear assignment problem are as follow:

Then transform the objective function by using first order Taylor series

\[
Z_k(x) = Z_k(x_{ij}) + \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} - x_{ij}^*) \frac{\partial Z_k(x_{ij})}{\partial x_{ij}}
\]

Find satisfactory \( x_{ij}^* \) by solving the problem into a single objective of assignment problem.

\[
Min Z = \sum_{k=1}^{3} \left( Z_k(x_{ij}^*) + \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{ij} - x_{ij}^*) \frac{\partial Z_k(x_{ij})}{\partial x_{ij}} \right)
\]

Subject to

\[
\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for} \quad i = 1, 2, \ldots, n
\]

\[
\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for} \quad j = 1, 2, \ldots, n
\]

\( x_{ij} = 0 \) or \( 1 \) \( \forall \ i, j \)

Step 5 Substitute the values of \( x_{ij}^* \) into the objective of FFAP, we get optimal fuzzy cost.

5.1 Numerical Example

To illustrate the proposed method, consider the following example of FFAP.

Example: Consider the following FFAP

\[
Min Z = \frac{(0,3,6)x_{11} + (1,4,7)x_{12} + (1,4.5,6)x_{21} + (3,4.5,8)x_{22}}{(0,2,4)x_{11} + (2,4,6)x_{12} + (1,4,7)x_{21} + (2,6.5,9)x_{22}}
\]

Subject to

\[
x_{11} + x_{12} = 1
\]

\[
x_{21} + x_{22} = 1
\]

\[
x_{11} + x_{21} = 1
\]

\[
x_{12} + x_{22} = 1
\]

\( x_{ij} = 0 \) or \( 1 \) \( \forall \ i, j \)

Solution : To solve the solution of above problem, the steps of proposed method, discussed in Section 5, are as follow:

Step 1:
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\[ \text{Min } Z = \frac{(x_{12} + 3x_{21} + 3x_{11} + 4x_{12} + 4.5x_{21} + 4.5x_{22}, 6x_{11} + 7x_{12} + 6x_{21} + 8x_{22})}{(2x_{12} + 2x_{21} + 2x_{22}, 2x_{11} + 4x_{12} + 4.5x_{21} + 4.5x_{22}, 6x_{11} + 7x_{12} + 6x_{21} + 9x_{22})} \]

Subject to
\[ x_{11} + x_{12} = 1 \]
\[ x_{21} + x_{22} = 1 \]
\[ x_{11} + x_{21} = 1 \]
\[ x_{12} + x_{22} = 1 \]
\[ x_{ij} = 0 \text{ or } 1 \forall \ i,j \]

Now the above problem can be reduced as
\[ \text{Min } Z = \left( \frac{x_{12} + 3x_{21} + 3x_{11} + 4x_{12} + 4.5x_{21} + 4.5x_{22}, 6x_{11} + 7x_{12} + 6x_{21} + 8x_{22}}{4x_{11} + 6x_{12} + 7x_{21} + 9x_{22}}, \frac{2x_{11} + 4x_{12} + 4.5x_{21} + 4.5x_{22}, 6x_{11} + 7x_{12} + 6x_{21} + 2x_{22}}{2x_{12} + 2x_{21} + 2x_{22}} \right) \]

Subject to
\[ x_{11} + x_{12} = 1 \]
\[ x_{21} + x_{22} = 1 \]
\[ x_{11} + x_{21} = 1 \]
\[ x_{12} + x_{22} = 1 \]
\[ x_{ij} = 0 \text{ or } 1 \forall \ i,j \]

Step 2 The above problem, obtained in Step 1, is equivalent to the following multi objective fractional assignment problem.

\[ \text{Min } Z_1 = \frac{x_{12} + 3x_{21} + 3x_{11} + 4x_{12} + 4.5x_{21} + 4.5x_{22}}{4x_{11} + 6x_{12} + 7x_{21} + 9x_{22}} \]
\[ \text{Min } Z_2 = \frac{3x_{11} + 4x_{12} + 4.5x_{21} + 4.5x_{22}}{2x_{11} + 4x_{12} + 4.5x_{21} + 6.5x_{22}} \]
\[ \text{Min } Z_3 = \frac{6x_{11} + 7x_{12} + 6x_{21} + 8x_{22}}{2x_{12} + 2x_{21} + 2x_{22}} \]

Subject to
\[ x_{11} + x_{12} = 1 \]
\[ x_{21} + x_{22} = 1 \]
\[ x_{11} + x_{21} = 1 \]
\[ x_{12} + x_{22} = 1 \]
\[ x_{ij} = 0 \text{ or } 1 \forall \ i,j \]

Step 3 Solving each objective function one by one we get
\[ Z_1^*(x_{11} = 0, x_{12} = 1, x_{21} = 1, x_{22} = 0), \]
\[ Z_2^*(x_{11} = 1, x_{12} = 0, x_{22} = 1, x_{21} = 0). \]
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Step 4 The multiobjective function obtained in Step 3, are transformed into single objective function by using 1st order Taylor’s series

\[ Z_1(x) = Z_1(x_{11} = 0, x_{12} = 1, x_{21} = 1, x_{22} = 0) + (x_{12} - 1) \frac{\partial Z_1}{\partial x_{12}} (x_{11} = 0, x_{12} = 1, x_{21} = 1, x_{22} = 0) + (x_{21} - 0) \frac{\partial Z_1}{\partial x_{21}} (x_{11} = 0, x_{12} = 1, x_{21} = 1, x_{22} = 0) \]

\[ \frac{\partial Z_1}{\partial x_{11}} = -4x_{12} - 4x_{21} - 12x_{22} \\
\frac{\partial Z_1}{\partial x_{12}} = 4x_{11} + 6x_{12} + 7x_{21} + 9x_{22} \]

\[ \frac{\partial Z_1}{\partial x_{21}} = 4x_{11} + 6x_{12} + 7x_{21} + 9x_{22} \]

\[ \frac{\partial Z_1}{\partial x_{22}} = 4x_{11} + 6x_{12} + 7x_{21} + 9x_{22} \]

\[ \frac{\partial Z_1}{\partial x_{11}} + \frac{\partial Z_1}{\partial x_{12}} + \frac{\partial Z_1}{\partial x_{21}} + \frac{\partial Z_1}{\partial x_{22}} = \frac{2}{13} + x_{11}(-\frac{8}{169}) + (x_{12} - 1)(-\frac{1}{169}) + x_{21} \frac{21}{169} \]

\[ Z_2(x) = \frac{1}{169}(-8x_{11} + x_{12} - x_{21} + 21x_{22} + 26) \]

Similarly

\[ Z_2(x) = Z_2(x_{11} = 1, x_{12} = 0, x_{21} = 0, x_{22} = 1) + (x_{11} - 1) \frac{\partial Z_2}{\partial x_{11}} + (x_{12} - 0) \frac{\partial Z_2}{\partial x_{12}} + (x_{21} - 0) \frac{\partial Z_2}{\partial x_{21}} + (x_{22} - 0) \frac{\partial Z_2}{\partial x_{22}} \]

\[ = \frac{1}{72.25} (10.5x_{11} + 4x_{12} + 8.25x_{21} - 10.5x_{22} + 63.75) \]

\[ Z_3(x) = Z_3(x_{11} = 0, x_{12} = 1, x_{21} = 1, x_{22} = 0) + (x_{11} - 0) \frac{\partial Z_3}{\partial x_{11}} + (x_{12} - 1) \frac{\partial Z_3}{\partial x_{12}} + (x_{21} - 0) \frac{\partial Z_3}{\partial x_{21}} + (x_{22} - 0) \frac{\partial Z_3}{\partial x_{22}} \]

\[ = \frac{1}{9} (18x_{11} - 5x_{12} + 6x_{21} - 2x_{22} + 38) \]

Now the above problem is converted into an equivalent single objective linear programming problem as:
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\[ Min \ Z = \frac{-8}{169} + \frac{10.5}{72.25} + 2x_{11} + \left( \frac{1}{169} + \frac{4}{72.25} - \frac{5}{9} \right)x_{12} + \left( \frac{-1}{169} + \frac{8.25}{72.25} + \frac{6}{9} \right)x_{21} \]

\[ + \left( \frac{21}{169} - \frac{10.5}{72.25} - \frac{2}{9} \right)x_{22} + \left( \frac{26}{169} + \frac{63.75}{72.25} + \frac{38}{9} \right) \]

(or)

\[ Min \ Z = 2.098x_{11} - 0.4943x_{12} + 0.775x_{21} - 0.2432x_{22} + 5.2584 \]

Subject to

\[ x_{11} + x_{12} = 1 \]
\[ x_{21} + x_{22} = 1 \]
\[ x_{11} + x_{21} = 1 \]
\[ x_{12} + x_{22} = 1 \]
\[ x_{ij} = 0 \text{ or } 1 \quad \forall \ i, j \]

After solving the above problem by using classical method, we get:

\[ x_{11} = 0, x_{12} = 1, x_{21} = 1, x_{22} = 0 \]

and \[ Min \ Z = 5.5391 \]

**Step 5** Substitute the obtained values of \( x_{ij} \) into the objective function of given problem, we get

\[ Min \ Z = \frac{(1,4,7) + (1,4,5,6)}{(2,4,6) + (1,4,7)} = \frac{(2,8,5,13)}{(3,8,13)} \]

**VI. CONCLUSION**

In this paper, a method for solving fuzzy fractional assignment problem is proposed. In the proposed method, FFAP is transformed to a MOLP of assignment problem and the resultant problem is converted into linear programming problem by using Taylor’s series method. The advantage of proposed method is that there is no need of ranking function to convert FFAP into linear programming problem.

**VII. ADVANTAGES OF PROPOSED METHOD OVER THE EXISTING METHODS OF ASSIGNMENT PROBLEM.**

In the existing methods of fuzzy assignment problems, ranking function was used to convert the fuzzy assignment problem into crisp assignment problem. But different ranking methods were used in the literature to convert fuzzy assignment problem into crisp assignment problem. Therefore because of different ranking methods, sometimes different results may arise. So it is not possible to find exact solution with this technique. But in the proposed method, FFAP is converted into linear programming problem by using Taylor’s series technique. In this method there is no need of ranking function to convert FFAP into linear assignment problem. Hence the proposed method is better than the existing methods.

**REFERENCES**


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