Limit Theorems on Fuzzy Markov Chains

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Abstract: In this paper we attempt to show the limit theorems for fuzzy Markov chains. Using stationary distribution we establish conditions for the existence of a Fuzzy Markov chain.

Key Words: Fuzzy Markov chain, Fuzzy Transition Probability and Fuzzy functions.

I. INTRODUCTION

Markov chains are one of the most important tools to model random phenomena evolving in time. A weak point of the most widely used model is that transition probabilities have to be constant and precisely known. An attempt to relax this restriction was proposed by Skulj[8] where the assumption of precisely known initial and transition probabilities is relaxed so that probability intervals are used instead of precise probabilities. Their model is based on the assumption that constant classical probabilities rule the process but only approximations are known instead of precise values.

The theory of Markov systems provide an effective and powerful tool for describing State of the system. Since numerous applied probability models can be adopted in their framework. Roughly speaking the Markov property requires that knowledge of the current state of the system provides all the information relevant to predicting its future. There have been a few other papers published on fuzzy Markov Chains[2,3,5,6].

The organization of the paper is as detailed below

Section 2 is devoted to fuzzy functions where Continuous we have defined the fuzzy functions.

Section 3 is addressing the notions of limit theorems on Fuzzy Markov chain

In Section 4 we are discussing about stationary distribution of a fuzzy Markov chain. We establish the conditions for the existence of a Markov chain.

II. FUZZY FUNCTIONS

Set valued functions and their calculus were found useful in of the problem in economics [1] and control theory [4]. From a probabilistic point of view random sets have a rather well developed theory [7].

M is a set, a fuzzy subset of M is a function u:M→[0,1]. The set of all fuzzy subsets of M, F(M) is a completely distribution lattice which includes the ordinary subsets of M. For any fuzzy subset u:M→[0,1] denote by Lα(u)={m∈M;u(m)≥α} α∈[0,1] is the α-level set of u.

If M is a vector space a fuzzy subset u∈F(M) is called a fuzzy Convex subset if u(λm1+(1-λ)m2)≥min[u(m1),u(m2)] for every m1, m2∈M, λ∈[0,1].

If X is a reflexive Banach space, in order to extend the Hausdorff distance we shall consider the subset F0(X) of F(X) containing all fuzzy sets u:X→[0,1] with properties

i) u is upper semi continuous.
ii) u is fuzzy convex.
iii) Lα(u) is compact for every α≠0.

If u,v∈F0(X) we define the distance between u and v by

d(u,v) = supα∈[0,1]dH(Lα(u),Lα(v)) Where dH denotes the Hausdorff distance.

Let X be a normed space, and u be an open subset of X. Let y be a reflexive Banach space. By a fuzzy function we mean a function F:u→F0(y) such a function associates to each point x∈U a fuzzy subset F(x) of y clearly such fuzzy functions generalizes set valued function u→Q(y).

III. LIMIT THEOREMS

LEMMA: 3.1

If the fuzzy states Fs is recurrent and Fs→Fr, then Fr is recurrent and fFsFr=fFrFs=1.

Proof:

Assume FssFr for otherwise there is nothing to prove.

Since fFsFr>0 there exists no such that P(>) > 0 and

P(>) = 0 for 0<m<n. (3.1)

Since P(>) > 0 we can find states F1,F2,..,F such that
\( P_{F_{F_{1}}} \cdots P_{F_{F_{n}}F_{1}} > 0 \) and none of the states \( F_{1}, F_{2}, \ldots, F_{n_{0}} \) equal \( F_{s} \) or \( F_{r} \), for if one of them did equal \( F_{s} \) or \( F_{r} \) it would be possible to go from \( F_{s} \) to \( F_{r} \) with positive probability in fewer then \( n_{0} \) steps in contradiction to (3.1).

Suppose \( f_{F_{F_{s}}F_{s}} < 1 \). Then a Markov chain starting from \( I \) has positive probability 1-\( f_{F_{F_{s}}F_{s}} \) of never hitting \( F_{s} \) and that implies it has positive probability \( P_{F_{F_{1}}F_{1}} \cdots P_{F_{F_{n_{0}}F_{1}}} (1 - f_{F_{F_{s}}F_{s}}) \) of visiting the states \( F_{r}F_{r} \cdots F_{r}F_{n_{0}} \), \( F_{r} \) successively in the first \( n_{0} \) steps and never return to \( F_{s} \) after \( n_{0} \) steps. But if this happens then the fuzzy Markov chain never return to \( F_{s} \) at any time \( n > 1 \) and that contradict the fact that \( F_{s} \) is recurrent. So \( f_{F_{F_{s}}F_{s}} = 1 \).

Since \( f_{F_{F_{s}}F_{s}} = 1 \) there exists \( n_{1} \) such that \( P_{F_{F_{s}}F_{s}}(n_{1}) > 0 \).

Now
\[
P_{F_{F_{r}}F_{r}}(n_{1} + n_{2}) \geq P_{F_{F_{s}}F_{s}}(n_{1})P_{F_{s}}(n_{2})P_{F_{F_{r}}}(n_{0})
\]
and hence
\[
\sum_{n=1}^{\infty} P_{F_{F_{r}}F_{r}}(n) \geq \sum_{n=1}^{\infty} P_{F_{F_{r}}F_{r}}(n_{1} + n_{2})
\]
\[
\geq P_{F_{F_{s}}F_{s}}(n_{1})P_{F_{s}}F_{r}(n_{2}) \sum_{n=1}^{\infty} P_{F_{s}}(n) = \infty
\]

Hence \( F_{r} \) is recurrent.

Since \( F_{s} \) is recurrent and \( F_{r} \rightarrow F_{s} \) (\( f_{F_{F_{r}}F_{s}} = 1 \)) from the first part of the proof it follows that \( f_{F_{F_{s}}F_{s}} = 1 \).

**THEOREM: 3.1**

\[
P_{F_{r}F_{s}}(n) = \sum_{m=1}^{\infty} f_{F_{r}F_{s}}(m)P_{F_{s}}(n-m) \text{ for all } m = 1, 2, \ldots, n
\]

Proof:

\[
P_{F_{r}F_{s}}(n) = \bigcup_{\alpha \in [0, 1]} \alpha(P_{|F_{r}F_{s}}(n))_{\alpha}
\]
\[
= \bigcup_{\alpha \in [0, 1]} \alpha(P[X_{n} = F_{s} | X_{0} = F_{r}])_{\alpha}
\]
\[
= \bigcup_{\alpha \in [0, 1]} \alpha P[X_{n} = (F_{s})_{\alpha} X_{0} = (F_{r})_{\alpha}]
\]
\[
= \sum_{m=1}^{n} \bigcup_{\alpha \in [0, 1]} \alpha P[X_{n} = (F_{s})_{\alpha} X_{m} = (F_{s})_{\alpha} X_{m-1} \neq (F_{s})_{\alpha} \ldots X_{1} \neq (F_{s})_{\alpha} | X_{0} = (F_{r})_{\alpha}]
\]

We take \( X_{m} = (F_{s})_{\alpha} A \), \( X_{m} = (F_{s})_{\alpha} X_{m-1} \neq (F_{s})_{\alpha} \ldots X_{1} \neq (F_{s})_{\alpha} = B_{m} \) and \( X_{0} = (F_{r})_{\alpha} = c \)

\[
P_{F_{r}F_{s}}(n) = \sum_{m=1}^{n} P[AB_{m} | c]
\]

Where \( B_{m} \) are disjoint and \( \bigcup_{m=1}^{n} B_{m} = A \)

Hence

\[
P_{F_{r}F_{s}}(n) = \sum_{m=1}^{n} \frac{P[AB_{m} | c]P[B_{m} | c]}{P[c]P[AB_{m} | c]}
\]
\[
= \sum_{m=1}^{n} \frac{P[|AB_{m} | c] | P[B_{m} | c]}{P[c]P[AB_{m} | c]}
\]
\[
= \sum_{m=1}^{n} \bigcup_{\alpha \in [0, 1]} \alpha P[X_{n} = (F_{s})_{\alpha} X_{m} = (F_{s})_{\alpha} X_{m-1} \neq (F_{s})_{\alpha} \ldots X_{1} \neq (F_{s})_{\alpha} | X_{0} = (F_{r})_{\alpha}]
\]
\[
\bigcup_{\alpha \in [0, 1]} \alpha P[X_{n} = (F_{s})_{\alpha} X_{m} = (F_{s})_{\alpha} X_{m-1} \neq (F_{s})_{\alpha} \ldots X_{1} \neq (F_{s})_{\alpha} | X_{0} = (F_{r})_{\alpha}]
\]
\[
= \sum_{m=1}^{\infty} \bigcup_{\alpha \in [0, 1]} \alpha P[X_{n} = (F_{s})_{\alpha} X_{m} = (F_{s})_{\alpha} f_{F_{F_{s}}F_{s}}(m)]
\]
\[\lim_{n \to \infty} \sum_{m=1}^{\infty} \alpha P_{F_S F_R}^{(n-m)} f_{F_R F_S}^{(m)}\]

\[= \sum_{m=1}^{\infty} \alpha P_{F_S F_R}^{(n-m)} f_{F_R F_S}^{(m)}\]

**Theorem 3.2 (Limit Theorem)**

Let \( F_S \) be a fixed state in a fuzzy Markov chain and \( F_R \) be an arbitrary state.

(i) If \( F_S \) is transient then \( P_{F_S F_R}^{(n)} \to 0 \) as \( n \to \infty \).

(ii) If \( F_S \) is null recurrent then \( P_{F_S F_R}^{(n)} \to 0 \).

(iii) If \( F_S \) is positive recurrent and the Markov chain is aperiodic then \( P_{F_S F_R}^{(n)} \to f_{F_S F_R}^{F_R F_S} \).

Proof:

By theorem 3.1

\[P_{F_R F_S}^{(n)} = \bigcup_{\alpha \in (0, 1]} \alpha (P_{F_R F_S}^{(n)})_{\alpha}\]

\[= \sum_{m=1}^{n} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(m)})_{\alpha} (P_{F_S F}^{(n-m)})_{\alpha}\]

\[= \sum_{m=1}^{n} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(m)})_{\alpha} (P_{F_S F}^{(n-m)})_{\alpha} + \sum_{m=n+1}^{n} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(m)})_{\alpha} (P_{F_S F}^{(n-m)})_{\alpha} \] \quad (3.2)

Where \( n < n' < n; (n \geq 1) \)

For \( \varepsilon > 0 \) take \( n' \) and \( n \) so large that

\[\sum_{m=n+1}^{n} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(m)})_{\alpha} < \varepsilon \] \quad (3.3)

When \( F_S \) is transient or null recurrent take \( n \) so large that

\[\bigcup_{\alpha \in (0, 1]} \alpha (P_{F_S F}^{(n-m)})_{\alpha} < \varepsilon \text{ for all } 0 \leq m < n' < n \]

By (3.2) and (3.3) we have

\[0 \leq \bigcup_{\alpha \in (0, 1]} \alpha (P_{F_R F_S}^{(n-m)})_{\alpha} - \sum_{m=n+1}^{n} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(m)})_{\alpha} (P_{F_S F}^{(n-m)})_{\alpha}\]

\[= \sum_{m=n+1}^{n} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(m)})_{\alpha} (P_{F_S F}^{(n-m)})_{\alpha}\]

\[\leq \sum_{m=n+1}^{n} \alpha (f_{F_R F_S}^{(m)})_{\alpha} < \varepsilon \] \quad (3.4)

\[0 \leq \lim_{n \to \infty} \bigcup_{\alpha \in (0, 1]} \alpha (P_{F_R F_S}^{(n)})_{\alpha}\]

\[\leq \varepsilon + \varepsilon \sum_{m=n+1}^{n} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(m)})_{\alpha} \quad \text{from (3.4)}\]

\[\leq \varepsilon + \varepsilon \sum_{m=n+1}^{n} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(m)})_{\alpha} = 2 \varepsilon \text{ for all } \varepsilon > 0\]

Therefore \( \bigcup_{\alpha \in (0, 1]} \alpha (P_{F_R F_S}^{(n)})_{\alpha} \to 0 \) as \( n \to \infty \).

(iii) Give that, the fuzzy state \( F_S \) is positive recurrent and the fuzzy Markov chain is aperiodic.

Take \( n \to \infty \) and \( n' \) fixed.

Then

\[0 \leq \lim_{n \to \infty} \bigcup_{\alpha \in (0, 1]} \alpha (P_{F_R F_S}^{(n)})_{\alpha} - \lim_{n \to \infty} \sum_{m=1}^{n'} \bigcup_{\alpha \in (0, 1]} \alpha (f_{F_R F_S}^{(n)})_{\alpha} (P_{F_S F}^{(n-m)})_{\alpha}\]

\[< \varepsilon \text{ By (3.4)}\]
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= \lim_{n \to \infty} \alpha (P_{FrFs}^{(n)})_a - \sum_{m=1}^{n'} \sum_{a \in (0,1)} \alpha (f_{FrFs}^{(n)})_a \frac{1}{\mu_{Fs}}

= \lim_{n \to \infty} \sum_{a \in (0,1)} \alpha (P_{FrFs}^{(n)})_a - \sum_{m=1}^{n'} \sum_{a \in (0,1)} \alpha (f_{FrFs}^{(n)})_a

\leq \epsilon

Take \( n' \to \infty \)

\[ 0 \leq \lim_{n \to \infty} \sum_{a \in (0,1)} \alpha (P_{FrFs}^{(n)})_a - \sum_{m=1}^{n'} \sum_{a \in (0,1)} \alpha (f_{FrFs}^{(n)})_a \frac{1}{\mu_{Fs}} \]

\( \alpha (P_{FrFs}^{(n)})_a \to \frac{1}{\mu_{Fs}} (f_{FrFs}^{(n)})_a \)

\[ \leq \epsilon \]

\( \alpha (P_{FrFs}^{(n)})_a \to \frac{1}{\mu_{Fs}} (f_{FrFs}^{(n)})_a \)

\[ \lim_{n \to \infty} \sum_{a \in (0,1)} \alpha (P_{FrFs}^{(n)})_a = \frac{1}{\mu_{Fs}} \]

\[ \mu_{Fs} \]

\[ \pi \]

\[ \pi = (\pi_1, \pi_2, \ldots) \]

\[ \lim_{n \to \infty} \sum_{a \in (0,1)} \alpha (P_{FrFs}^{(n)})_a = \pi_{Fs} \geq 0 \text{ for all } Fr \geq 1. \]

\[ \text{Then } \pi \text{ is called the steady state distribution of the Markov chain with Transition matrix } (P_{FrFs}). \]

**Theorem 4.1**

Let a Fuzzy Markov chain is irreducible, aperiodic and positive. Then

(i) \[ \lim_{n \to \infty} P_{FrFs}^{(n)} = \pi_{Fr} \]

(ii) \[ \pi_{Fs} > 0 \sum_{FS} \pi_{Fs} = 1 \]

(iii) \[ \pi_{Fs} = \sum_{FS} \pi_{Fs} P_{Fs} \]

More over (ii) and (iii) determine \( \{\pi_{Fs}, FS \in S\} \) Completely.

\[ \text{Proof:} \]

(i) The Proof of (i) follows from theorem 2.2 and the lemma.

(ii) \[ \pi_{Fs} = \frac{1}{\mu_{Fs}} > 0 \]

Suppose \( S_M \) is a subset of the state space \( S \) with exactly \( M \) states.

Now,

\[ \sum_{FS \in S_M} P_{FrFs}^{(n)} \]

**IV. STATIONARY DISTRIBUTION**

**Definition: 4.1**

A probability distribution is \( \{V_{Fs}\} \) with \( V_{Fs} \geq 0 \sum_{FS} V_{Fs} = 1 \) is called a stationary distribution for a Markov chain with transition matrix \( P_{FrFs} \) if

\[ V_{Fs} = \sum_{Fr} V_{Fs} P_{FrFs} \]

\[ = \sum_{Fr} \sum_{a \in (0,1)} \alpha (P_{FrFs} )_a \]

\[ = \sum_{Fr} \sum_{Fr} V_{Fs} \sum_{a \in (0,1)} \alpha (P_{FrFr} )_a (P_{FrFs} )_a \]

\[ = \sum_{Fr} V_{Fs} \sum_{Fr} \sum_{a \in (0,1)} \alpha (P_{FrFr} )_a (P_{FrFs} )_a \]

\[ = \sum_{Fr} V_{Fs} \sum_{Fr} \sum_{a \in (0,1)} \alpha (P_{FrFs}^{(2)})_a \]

\[ \ldots \]

\[ = \sum_{Fr} V_{Fs} \sum_{Fr} \sum_{a \in (0,1)} (P_{FrFs} )_a \]

\[ = \sum_{Fr} V_{Fs} P_{FrFs}^{(n)} \]

Suppose a stationary distribution \( \pi = (\pi_1, \pi_2, \ldots) \) exists. Also suppose

\[ \lim_{n \to \infty} \sum_{a \in (0,1)} \alpha (P_{FrFs}^{(n)})_a = \pi_{Fs} \geq 0 \text{ for all } Fr \geq 1. \]

Then \( \pi \) is called the steady state distribution of the Markov chain with Transition matrix \( (P_{FrFs}). \)
\[ \sum_{F_s \in S_M} \alpha (P_{FrFs} (n))_a \leq \sum_{F_s \in S} \alpha (P_{FrFs} (n))_a = 1 \]

Let \( n \to \infty \) then
\[ \sum_{F_s \in S_M} \pi_{Fs} \leq 1 \]
Then taking limit \( M \to \infty \)
\[ \sum_{F_s \in S_M} \pi_{Fs} \leq 1 \quad (4.1) \]

Then taking limit
\[ \sum_{F_s \in S_M} \sum_{a \in [0,1]} \alpha (P_{FrFk} (n))_a (P_{FkF} (n))_a \]
\[ \leq \sum_{F_s \in S_M} \sum_{a \in [0,1]} \alpha (P_{FrFk} (n))_a P_{FkFs} \]
\[ = \sum_{a \in [0,1]} \alpha (P_{FrFs} (n+1))_a \]
Let \( n \to \infty \) then
\[ \lim_{n \to \infty} \sum_{a \in [0,1]} \alpha (P_{FrFs} (n+1))_a = \sum_{F_s \in S_M} \pi_{Fk} \cup \sum_{a \in [0,1]} \alpha (P_{FkF} (n))_a \leq \pi_{Fs} \]
Then letting \( M \to \infty \) we get
\[ \sum_{F_s \in S_M} \pi_{Fk} \cup \sum_{a \in [0,1]} \alpha (P_{FkF} (n))_a \leq \pi_{Fs} \quad (4.2) \]
\[ \sum_{F_s \in S} \pi_{Fs} \cup \sum_{a \in [0,1]} \alpha (P_{FsFr} (n+2))_a \]
\[ = \sum_{F_s \in S} \pi_{Fs} \cup \sum_{a \in [0,1]} \alpha (P_{FsFk} (n))_a (P_{FkFr} (n))_a \]
\[ = \sum_{a \in [0,1]} \sum_{F_s \in S} \pi_{Fs} \alpha (P_{FsFk} (n))_a (P_{FkFr} (n))_a \]
\[ \leq \sum_{a \in [0,1]} \sum_{F_s \in S} \alpha (P_{FkFr} (n))_a \]
\[ = \sum_{F_s \in S} \pi_{Fk} P_{FkFr} \leq \pi_{Fr} \]
By induction
\[ \sum_{F_s \in S} \pi_{Fs} \cup \sum_{a \in [0,1]} \alpha (P_{FsFr} (n))_a \leq \pi_{Fr} \text{ for all } n \geq 1; Fs \in S \]
Now
\[ \pi_{Fk} = \pi_{Fk} \left( \sum_{F_s \in S} P_{FkFs} (n) \right) \]
\[ \left( \sum_{F_s \in S} P_{FkFs} (n) = 1 \right) \]
$$\sum_{F_k \in S} \pi_{F_k} = \sum_{F_k \in S} \sum_{F_s \in S} \alpha \pi_k \left( P_{F_k F_s}^{(n)} \right)_\alpha$$

$$= \sum_{F_k \in S} \sum_{F_s \in S} \pi_k \alpha \left( P_{F_k F_s}^{(n)} \right)_\alpha$$

By Fubinis theorem.

Suppose

$$\sum_{F_k \in S} \pi_{F_k} \bigcup_{a \in (0,1)} \alpha P_{F_k F_s}^{(n)} < \pi_{F_s}$$

Then

$$\sum_{F_k \in S} \sum_{F_s \in S} \pi_k \bigcup_{a \in (0,1)} \alpha \left( P_{F_k F_s}^{(n)} \right)_\alpha < \sum_{F_s \in S} \pi_{F_s}$$

$$\sum_{F_k \in S} \pi_{F_k} < \sum_{F_s \in S} \pi_{F_s}$$

Which is a Contradiction.

Thus

$$\sum_{F_k \in S} \alpha \left( P_{F_k F_s}^{(n)} \right)_\alpha = \sum_{F_k \in S} \pi_{F_k} \pi_{F_k F_s}^{(n)}$$

$$= \pi_{F_s} \text{ for } n \geq 1 \quad (4.3)$$

In particular for

$$n \geq 1 \quad \sum_{F_s \in S} \pi_{F_s} P_{F_s F_r} = \pi_{F_r}$$

This proves (iii).

Moreover by Lebesgue Dominated convergence theorem and part (i) letting $n \to \infty$ in (4.3)

$$\sum_{F_s \in S} \pi_{F_s} \pi_{F_r} = \pi_{F_r}$$

Now $\pi_{F_r} > 0$ that gives

$$\sum_{F_s \in S} \pi_{F_s} = 1$$

To show that the solution given by (ii) and (iii) is unique. Suppose that $\{x_{F_r}, F_r \in S\}$ is another such solution satisfying $x_{F_r} > 0$

$$\sum_{F_s \in S} \pi_{F_s} = 1$$

and

$$x_{F_r} = \sum_{F_s \in S} x_{F_s} P_{F_s F_r}$$

$$= \sum_{F_s \in S} x_{F_s} \bigcup_{a \in (0,1)} \alpha \left( P_{F_s F_r} \right)_\alpha$$

$$= \sum_{F_s \in S} \left( \sum_{F_k \in S} \bigcup_{a \in (0,1)} \alpha x_k \left( P_{F_k F_r} \right)_\alpha \right) \left( P_{F_s F_r} \right)_\alpha$$

$$= \sum_{F_k \in S} \left( \sum_{F_s \in S} \bigcup_{a \in (0,1)} \alpha \left( P_{F_k F_s} \right)_\alpha \left( P_{F_s F_r} \right)_\alpha \right)$$

$$= \sum_{F_k \in S} x_{F_k} P_{F_k F_r}^{(n)}$$

(By Fubinis theorem)

$$= \sum_{F_s \in S} x_{F_s} P_{F_s F_r}^{(n)}$$

$$= \sum_{F_s \in S} x_{F_s} P_{F_s F_r}^{(n)}$$

$$= \sum_{F_s \in S} x_{F_s} P_{F_s F_r}^{(n)}$$

By the Lebesgue Dominated Convergence theorem, Letting $n \to \infty$

$$x_{F_r} = \sum_{F_k \in S} x_{F_k} \pi_{F_r} = \pi_{F_r} \sum_{F_k \in S} x_{F_k} = \pi_{F_r} \text{ for all } F_r \in S$$

Thus the solution $\{\pi_i, i \in S\}$ is unique.
THEOREM: 4.2
A Fuzzy Markov chain remains Markov if time is reversed.

\[ P(X_{n-1} = F_{n-1} | X_n = F_n, X_{n+1} = F_{n+1}, \ldots) = \bigcup_{\alpha \in (0,1]} \alpha P(X_{n-1} = (F_{n-1})_{\alpha} | X_n = (F_n)_{\alpha}, X_{n+1} = (F_{n+1})_{\alpha}, \ldots) \]

Proof:

\[ P(X_{n-1} = F_{n-1} | X_n = F_n, X_{n+1} = F_{n+1}, \ldots) = \bigcup_{\alpha \in (0,1]} \alpha P(X_{n-1} = (F_{n-1})_{\alpha} | X_n = (F_n)_{\alpha}, X_{n+1} = (F_{n+1})_{\alpha}, \ldots) \]

\[ = \bigcup_{\alpha \in (0,1]} \alpha P(X_{n+1} = (F_{n+1})_{\alpha}, X_{n+2} = (F_{n+2})_{\alpha}, \ldots | X_n = (F_n)_{\alpha}, X_{n-1} = (F_{n-1})_{\alpha}) \]

\[ = \bigcup_{\alpha \in (0,1]} \alpha P(X_{n+1} = (F_{n+1})_{\alpha}, X_{n+2} = (F_{n+2})_{\alpha}, \ldots | X_n = (F_n)_{\alpha}, X_{n-1} = (F_{n-1})_{\alpha}) \]

\[ = P(X_{n-1} = (F_{n-1})_{\alpha} | X_n = (F_n)_{\alpha}) \]

\[ = P(X_{n-1} = (F_{n-1})_{\alpha} | X_n = (F_n)_{\alpha}) \]

THEOREM: 4.3
In a Fuzzy Markov chain if the present is specified then the past is independent of the future in the following sense.

\[ P(X_n = F_n, X_{n+k} = F_k | X_m = F_m) = P(X_n = (F_n)_{\alpha}, X_m = (F_m)_{\alpha})P(X_k = (F_k)_{\alpha} | X_m = (F_m)_{\alpha}) \]

Proof:

By the Chain rule of conditional probabilities

\[ P(X_n = F_n, X_{n+k} = F_k | X_m = F_m) = \bigcup_{\alpha \in (0,1]} \alpha P(X_n = (F_n)_{\alpha}, X_k = (F_k)_{\alpha} | X_m = (F_m)_{\alpha}) \]

\[ = \bigcup_{\alpha \in (0,1]} \alpha P(X_n = (F_n)_{\alpha}, X_k = (F_k)_{\alpha} | X_m = (F_m)_{\alpha}) \]

\[ = \bigcup_{\alpha \in (0,1]} \alpha P(X_n = (F_n)_{\alpha}, X_k = (F_k)_{\alpha} | X_m = (F_m)_{\alpha}) \]

\[ = \bigcup_{\alpha \in (0,1]} \alpha P(X_n = (F_n)_{\alpha}, X_k = (F_k)_{\alpha} | X_m = (F_m)_{\alpha}) \]

\[ = \bigcup_{\alpha \in (0,1]} \alpha P(X_n = (F_n)_{\alpha}, X_k = (F_k)_{\alpha} | X_m = (F_m)_{\alpha}) \]

REFERENCES


